THE STRAIN OF A GRAVITATING SPHERE OF VARIABLE DENSITY AND ELASTICITY*

BY

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I. Introductory. Statement of the problem

The problem of determining the strain of a gravitating elastic sphere from an initial condition of hydrostatic equilibrium, under the action of small disturbing forces having a spherical harmonic potential, has been solved for the case in which the density and elastic moduli are uniform throughout the body. It has also been solved for cases in which the density varies with the distance from the center, but with the restriction that the modulus of compression is infinite while the modulus of rigidity has a uniform value.

This problem is of especial interest because of its relation to estimates of the rigidity of the earth. Such estimates are based upon the comparison of the actual yielding of the earth to tidal and centrifugal forces (as inferred from certain refined observations) with the computed yielding of an elastic sphere having the same size and mass as the earth. For the purpose of such a comparison it is obviously desirable that the ideal sphere assumed in the computations should agree with the actual earth-sphere as closely as possible in all the elements essential to the problem. Now it is quite certain that the density of the earth varies greatly with distance from the center, that the material of the earth is far from incompressible, and that the earth as a whole is very much more rigid than the surface rocks. Our knowledge of the actual properties of the earth is much better represented by the assumption that all three of the quantities, density, modulus of compression, and modulus of rigidity are functions of distance from the center than by any of the more restricted assumptions which are essential to the above-mentioned solutions of the sphere problem. It is therefore desirable to solve, in as general a form as possible, the following problem:

A sphere of isotropic material, in which the density and elastic moduli are functions of distance from the center, and which would be in hydrostatic equilibrium under self-gravitation alone, is strained by small disturbing forces

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having a potential which is a spherical harmonic function of the coördinates and a simple harmonic function of the time; it is required to determine the strain.

The analysis given in the present paper leads to a solution of this problem which, though not fully general, is a considerable advance in generality over the restricted solutions above mentioned. The solution of the general problem is shown to depend upon that of a pair of simultaneous ordinary differential equations; and it is found that these may be solved in terms of convergent series when certain restrictions are imposed upon the functions expressing the density and elastic moduli. Moreover, these restrictions are not such as to detract seriously from the geodynamical interest of the solution.

The scope and order of the paper may be summarized as follows:

I. Introductory. Statement of the problem.

II. Establishment of fundamental equations in general form. The density and elastic moduli are here treated as unrestricted functions of distance from center, and the disturbing potential as a simple harmonic function of the time.

III. Solution of the equations when the density is variable while the elastic moduli are constants.* The density function is assumed to be rational and integral.

IV. Solution of equations when the density and both elastic moduli are variables. The assumption as to density is the same as in III, while the elastic moduli are represented by rational integral functions of a restricted form.

V. Particular solutions. Particular forms of the functions expressing density and elastic moduli are substituted in the formulas expressing the general solution given in IV, reducing these to the working formulas used in numerical computations.

VI. Application to the earth. Numerical results are given for the strain of a sphere having the size and mass of the earth, showing the effect of different assumptions regarding density, elasticity, and compressibility. Besides the results for the static problem, a series is also given for the case in which the disturbing potential is a simple harmonic function of the time.

II. Establishment of the fundamental equations

1. **Physical theory.** The analysis which follows is a generalization of that used in a former paper dealing with a less general form of the sphere problem,†

* The assumptions of variable density and variable elasticity are found to be independent of each other as regards their effect on the fundamental equations, and it seems advantageous to present separately the two parts of the analysis leading to the most general form of the equations which express the solution.

† *The strain of a gravitating, compressible elastic sphere*, these Transactions, vol. 11 (1910), pp. 203–248.

but with an amendment to the physical theory. This amendment leads to certain changes in the fundamental equations; but though physically important, these changes are algebraically of minor effect, leaving the same general method of solution still applicable.*

The problem of the strain of an elastic solid from a condition of great initial stress cannot be treated by the equations of elastic deformation in the ordinary form, since these equations assume the proportionality of actual stress to strain. Some of the earlier papers on the strain of a gravitating sphere treated the problem as if the body were strained from an initial "natural" configuration in which gravitation did not act. Applied to a body of planetary dimensions such a solution leads to deformations far beyond the elastic limits of known materials; but even apart from this fact, the basal assumption can have no validity as applied to actual bodies. It is only on the assumption of incompressibility that this method of solution leads to valid results. If the material is incompressible the computed effect of gravitation alone is a condition of hydrostatic stress throughout the sphere, with zero strain; so that the determination by this method of the strain due to small disturbing forces combined with gravitation amounts to the same thing as the determination of the strain due to the disturbing forces alone from an initial condition of hydrostatic stress.† For a compressible material, however, the reasoning fails.

It is reasonable to suppose that a solid body is elastic for small deformations even if initially in a condition of great stress. It seems reasonable, also, to apply the ordinary laws of elasticity to such a case, with this change: the relation "stress is proportional to strain" is replaced by the relation "increment of stress is proportional to strain."‡

* The defect in the physical theory as formerly stated was pointed out by the writer in a footnote to a later paper, these Transactions, vol. 11 (1910), p. 504. The same point was discussed by A. E. H. Love in his work *Some problems of geodynamics* (1911), in which the amended theory was applied in an able analysis of the sphere problem in the case of uniform density.

† This method was employed by G. Herglotz in a paper dealing with the strain of an incompressible sphere of variable density. *Über die Elastizität der Erde bei Berücksichtigung ihrer variablen Dichte*, Zeitschrift für Mathematik und Physik, vol. 52 (1905), p. 275. In this paper the solution for the general case of density varying with distance from the center is reduced to the solution of a linear ordinary differential equation of the sixth order whose coefficients depend upon the law of density, and the complete solution is given for certain cases of that law. When the assumption of incompressibility is introduced the analysis given in the present paper leads to a differential equation identical with that obtained by Herglotz.

‡ In the application of this principle in the writer's former paper cited above, "increment of stress" was computed by considering a definite volume-element before and during the strain. This was later seen to be an error, because the displacement of an element of matter causes a change in the gravitational force acting on that element which is of the same order of importance as the disturbing force. The reasonable method is rather to compare the stress-condi-

The differential equations of strain, formed in accordance with this principle, involve the initial stresses; these are in general indeterminate, and the solution of the equations is therefore indeterminate unless something is assumed regarding the initial stress-condition. The assumption which has commonly been made, and which will be made in the analysis herein given, is that the initial condition is that of hydrostatic equilibrium. This assumption is not only especially simple but is especially interesting in the geodynamical problem, since in the case of a body of planetary dimensions like the earth the predominating stress throughout the greater part of the body is without doubt a normal pressure.*

2. **Differential equations of strain of an isotropic elastic solid from a condition of initial stress.** As a preliminary to the discussion of the problem of the sphere it is useful to establish the fundamental equations in as general a form as possible. We therefore consider first the general case of an isotropic elastic solid undergoing a small strain from any state of initial stress; the fundamental assumption being that *increment of stress* is connected with strain by the same relations which, in the ordinary theory, are assumed to connect *actual stress* with strain.

In establishing the differential equations it is convenient to use rectangular coördinates; polar coördinates are however employed in the application to the sphere.† The notation for displacements, strains, and stresses will for the most part be that adopted in A. E. H. Love's treatise *The Mathematical Theory of Elasticity*, second edition; axial components of displacement are, however, denoted by u_x, u_y, u_z instead of u, v, w.

The equations of motion for an element of the body are

$$
\rho\frac{\partial^2 u_x}{\partial t^2} = \rho X + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z},
$$

(1)
$$
\rho\frac{\partial^2 u}{\partial t^2} = \rho Y + \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z},
$$

$$
\rho\frac{\partial^2 u_z}{\partial t^2} = \rho Z + \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z}.
$$

These equations are independent of elastic theory.

tion of an individual element of matter before and during strain. This is the amendment to the physical theory referred to above. The defective theory had also been employed in a paper by A. E. H. Love, *The gravitational stability of the earth*, Philosophical Transactions of the Royal Society, London, A, vol. 207 (1907), p. 171.

* This method of treating the geodynamical problem seems to have been first suggested by Lord Rayleigh. See paper *On the dilatational stability of the earth*, Proceedings of the Royal Society, London, A, vol. 77 (1906).

† Attention is called to the fact that after the change to polar coördinates the letters x, y, z are given new meanings.

The ordinary stress-strain relations for an isotropic elastic solid are expressed by the six equations

$$
\begin{gathered}
X_x = \lambda\Delta + 2\mu\frac{\partial u_x}{\partial x}, \qquad Y_y = \lambda\Delta + 2\mu\frac{\partial u_y}{\partial y}, \qquad Z_z = \lambda\Delta + 2\mu\frac{\partial u_z}{\partial z}, \\
Y_z = Z_y = \mu\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right), \qquad Z_x = X_z = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right), \\
X_y = Y_x = \mu\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right).
\end{gathered} \tag{2}
$$

Equations (2) apply to the case in which the body is strained from its " natural " state, i.e., from a condition of zero stress.

Let it now be supposed that the body is initially in equilibrium under the action of any bodily and surface forces, so that initial stresses exist throughout the material, and that u_x, u_y, u_z denote displacement *from this initial configuration*; it will be assumed that equations like (2) hold with the substitution for each stress-component of *the increment of that stress-component due to the strain.*

Let $\delta X_x, \delta Y_y, \cdots$ denote these increments of the stress-components for an element of material initially at (x, y, z); then the assumption is expressed by the equations

$$
\begin{gathered}
\delta X_x = \lambda\Delta + 2\mu\frac{\partial u_x}{\partial x}, \qquad \delta Y_y = \lambda\Delta + 2\mu\frac{\partial u_y}{\partial y}, \qquad \delta Z_z = \lambda\Delta + 2\mu\frac{\partial u_z}{\partial z}, \\
\delta Y_z = \delta Z_y = \mu\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right), \qquad \delta Z_x = \delta X_z = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right), \\
\delta X_y = \delta Y_x = \mu\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right).
\end{gathered} \tag{3}
$$

These equations give, by differentiation and addition,

$$
\begin{aligned}
\frac{\partial}{\partial x}\delta X_x + \frac{\partial}{\partial y}\delta X_y + \frac{\partial}{\partial z}\delta X_z &= (\lambda + \mu)\frac{\partial\Delta}{\partial x} + \mu\nabla^2 u_x \\
&\quad + \frac{\partial\lambda}{\partial x}\Delta + 2\frac{\partial\mu}{\partial x}\frac{\partial u_x}{\partial x} + \frac{\partial\mu}{\partial y}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) + \frac{\partial\mu}{\partial z}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right),
\end{aligned} \tag{4}
$$

with two similar equations which may be written by cyclic permutation of x, y, z.

Let the components of actual stress at the point (x, y, z) before strain be $X_x, X_y, \cdots$; and let $X'_x, X'_y, \cdots$ denote the same quantities *at the same point* during strain. Then the increment of stress for an individual element of

material initially at (x, y, z) is given by equations like the following:

$$\delta X_x = X'_x - X_x + u_x \frac{\partial X_x}{\partial x} + u_y \frac{\partial X_x}{\partial y} + u_z \frac{\partial X_x}{\partial z}; \tag{5}$$

and these values substituted in (4) reduce the first member to the expression

$$\begin{aligned} &\left(\frac{\partial X'_x}{\partial x} + \frac{\partial X'_y}{\partial y} + \frac{\partial X'_z}{\partial z}\right) - \left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\right) \\ &+ \frac{\partial}{\partial x}\left(u_x \frac{\partial X_x}{\partial x} + u_y \frac{\partial X_x}{\partial y} + u_z \frac{\partial X_x}{\partial z}\right) + \frac{\partial}{\partial y}\left(u_x \frac{\partial X_y}{\partial x} + u_y \frac{\partial X_y}{\partial y} + u_z \frac{\partial X_y}{\partial z}\right) \\ &+ \frac{\partial}{\partial z}\left(u_x \frac{\partial X_z}{\partial x} + u_y \frac{\partial X_z}{\partial y} + u_z \frac{\partial X_z}{\partial z}\right). \end{aligned} \tag{6}$$

But the first of equations (1) gives

$$\begin{aligned} &\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = -\rho X, \\ &\frac{\partial X'_x}{\partial x} + \frac{\partial X'_y}{\partial y} + \frac{\partial X'_z}{\partial z} = -\rho' X' + \rho' \frac{\partial^2 u_x}{\partial t^2}, \end{aligned} \tag{7}$$

in which accented letters denote values during strain and unaccented letters corresponding values before strain at the same point (x, y, z). Also

$$\rho' = \rho(1 - \Delta) - \left(u_x \frac{\partial \rho}{\partial x} + u_y \frac{\partial \rho}{\partial y} + u_z \frac{\partial \rho}{\partial z}\right); \tag{8}$$

and since ρ, X may replace ρ', X' in terms containing u_x, u_y, or u_z, the expression (6) becomes

$$\begin{aligned} &\rho\left(\frac{\partial^2 u_x}{\partial t^2} + X - X' + X\Delta\right) + X\left(u_x \frac{\partial \rho}{\partial x} + u_y \frac{\partial \rho}{\partial y} + u_z \frac{\partial \rho}{\partial z}\right) \\ &+ \frac{\partial}{\partial x}\left(u_x \frac{\partial X_x}{\partial x} + u_y \frac{\partial X_x}{\partial y} + u_z \frac{\partial X_x}{\partial z}\right) + \frac{\partial}{\partial y}\left(u_x \frac{\partial X_y}{\partial x} + u_y \frac{\partial X_y}{\partial y} + u_z \frac{\partial X_y}{\partial z}\right) \\ &+ \frac{\partial}{\partial z}\left(u_x \frac{\partial X_z}{\partial x} + u_y \frac{\partial X_z}{\partial y} + u_z \frac{\partial X_z}{\partial z}\right). \end{aligned} \tag{9}$$

Equation (4), with (9) substituted for the first member, is one of three differential equations applicable to the general case of strain of an isotropic elastic solid from a condition of initial stress. The six stress-components whose derivatives appear in these equations are connected by the relations obtained from (1) (with the acceleration terms omitted); but the solution will be indeterminate unless something more is known concerning the initial stresses.

3. **Simplification of equations when initial stress is hydrostatic.** It will hereafter be assumed that the initial stress-condition is that of hydrostatic equilibrium, so that all terms involving tangential stress-components vanish, while $X_x = Y_y = Z_z$. Using (1), the expression (9) may now be written

$$\begin{aligned}(10)\quad \rho\left(\frac{\partial^2 u_x}{\partial t^2} + X - X' + X\Delta\right) &+ X\left(u_x\frac{\partial\rho}{\partial x} + u_y\frac{\partial\rho}{\partial y} + u_z\frac{\partial\rho}{\partial z}\right)\\ &- \frac{\partial}{\partial x}(\rho u_x X + \rho u_y Y + \rho u_z Z).\end{aligned}$$

4. **Form assumed by equations when the body is a gravitating sphere.** Let it now be assumed that the body is a sphere, and that ρ, λ, and μ vary only with r, the distance from the center. If V denotes the gravitation potential before strain (a function of r only),

$$(11)\qquad X = \frac{x}{r}\frac{dV}{dr}, \qquad Y = \frac{y}{r}\frac{dV}{dr}, \qquad Z = \frac{z}{r}\frac{dV}{dr}.$$

Hence if u_r denotes radial displacement,

$$(12)\qquad u_x X + u_y Y + u_z Z = u_r\frac{dV}{dr};$$

$$\begin{aligned}(13)\quad \frac{\partial}{\partial x}(\rho u_x X + \rho u_y Y + \rho u_z Z) &= \frac{\partial}{\partial x}\left(\rho u_r\frac{dV}{dr}\right)\\ &= \rho\frac{\partial}{\partial x}\left(u_r\frac{dV}{dr}\right) + u_r\frac{x}{r}\frac{d\rho}{dr}\frac{dV}{dr}.\end{aligned}$$

Also

$$(14)\qquad X\left(u_x\frac{\partial\rho}{\partial x} + u_y\frac{\partial\rho}{\partial y} + u_z\frac{\partial\rho}{\partial z}\right) = u_r\frac{x}{r}\frac{d\rho}{dr}\frac{dV}{dr}.$$

Assuming now that the bodily forces during strain are derived from a potential V', the expression (10) reduces to the following:

$$(15)\qquad \rho\left[\frac{\partial^2 u_x}{\partial t^2} + \frac{\partial V}{\partial x}\Delta - \frac{\partial}{\partial x}\left(V' - V + u_r\frac{dV}{dr}\right)\right].$$

Substituting this for the first member of (4), and writing E_x for the terms in the second member which contain derivatives of λ and μ, we obtain the following as one of the three differential equations for the strain of a gravitating sphere:

$$(16)\quad (\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 u_x+E_x = \rho\left[\frac{\partial^2 u_x}{\partial t^2}+\frac{\partial V}{\partial x}\Delta-\frac{\partial}{\partial x}\left(V'-V+u_r\frac{dV}{dr}\right)\right].$$

The other two equations are obtained by cyclic permutation of x, y, z.

The first members of these equations are the same as would be obtained if there were no initial stress. The equations are in fact identical with those which would apply to strain from zero-stress under the action of forces whose axial components (per unit mass) are

$$(17)\quad \frac{\partial}{\partial x}\left(V'-V+u_r\frac{dV}{dr}\right)-\frac{\partial V}{\partial x}\Delta,\qquad \frac{\partial}{\partial y}\left(V'-V+u_r\frac{dV}{dr}\right)-\frac{\partial V}{\partial y}\Delta,$$
$$\frac{\partial}{\partial z}\left(V'-V+u_r\frac{dV}{\partial r}\right)-\frac{\partial V}{dz}\Delta.$$

In other words the strain may be treated as if due to three sets of bodily forces:

(*a*) Forces having potential $V'-V$; these include the disturbing force and the change in the gravitational force due to the changed configuration of the attracting mass.

(*b*) Forces having potential $u_r(dV/dr)$.

(*c*) The increment of gravitational force per unit volume due to the change $-\rho\Delta$ in the density of the attracted element.

5. **Equations in polar coördinates for gravitating sphere.** We now pass to polar coördinates, replacing x, y, z by r, θ, ϕ as independent variables, and u_x, u_y, u_z by u_r, u_θ, u_ϕ as component displacements, according to the following scheme of direction cosines.

	r	θ	ϕ
x	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
y	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
z	$\cos\theta$	$-\sin\theta$	0

Each of the required equations is obtained by multiplying the three equations like (16) by the proper direction cosines and adding. The second members of the resulting equations may be written by inspection (noticing that, to the first order of small quantities, the acceleration components are $\partial^2 u_r/\partial t^2$, $\partial^2 u_\theta/\partial t^2$, $\partial^2 u_\phi/\partial t^2$). The first members, aside from the terms obtained from E_x, E_y, E_z, are given in treatises dealing with the case in which λ and μ are constants.* Let the terms involving derivatives of λ and μ

* See A. E. H. Love's *Treatise on the Mathematical Theory of Elasticity*, second edition (1906), p. 138. For the values of ϖ_r, ϖ_θ, ϖ_ϕ, Δ, see p. 56.

be E_r, E_θ, E_ϕ; then the three equations in polar coördinates are

$$(18)\quad \begin{aligned}
&(\lambda + 2\mu)\frac{\partial \Delta}{\partial r} - \frac{2\mu}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\varpi_\phi \sin\theta) - \frac{\partial \varpi_\theta}{\partial\phi}\right] + E_r \\
&\qquad = \rho\left[\frac{\partial^2 u_r}{\partial t^2} + \frac{\partial V}{\partial r}\Delta - \frac{\partial}{\partial r}\left(V' - V + u_r\frac{dV}{dr}\right)\right], \\
&\frac{\lambda + 2\mu}{r}\frac{\partial \Delta}{\partial\theta} - \frac{2\mu}{r\sin\theta}\left[\frac{\partial \varpi_r}{\partial\phi} - \frac{\partial}{\partial r}(r\varpi_\phi \sin\theta)\right] + E_\theta \\
&\qquad = \rho\left[\frac{\partial^2 u_\theta}{\partial t^2} - \frac{1}{r}\frac{\partial}{\partial\theta}\left(V' - V + u_r\frac{dV}{dr}\right)\right], \\
&\frac{\lambda + 2\mu}{r\sin\theta}\frac{\partial \Delta}{\partial\phi} - \frac{2\mu}{r}\left[\frac{\partial}{\partial r}(r\varpi_\theta) - \frac{\partial \varpi_r}{\partial\theta}\right] + E_\phi \\
&\qquad = \rho\left[\frac{\partial^2 u_\phi}{\partial t^2} - \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(V' - V + u_r\frac{dV}{dr}\right)\right].
\end{aligned}$$

Here $\varpi_r, \varpi_\theta, \varpi_\phi$ are rotational components of strain.

It remains to determine E_r, E_θ, E_ϕ by carrying out the process above outlined. If λ and μ are unrestricted functions of the coördinates, the resulting values are

$$(19)\quad \begin{aligned}
E_r &= \Delta\frac{\partial\lambda}{\partial r} + 2\frac{\partial\mu}{\partial r}\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial\mu}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \\
&\quad + \frac{1}{r\sin\theta}\frac{\partial\mu}{\partial\phi}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right), \\
E_\theta &= \frac{\Delta}{r}\frac{\partial\lambda}{\partial\theta} + \frac{\partial\mu}{\partial r}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) + \frac{2}{r}\frac{\partial\mu}{\partial\theta}\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right) \\
&\quad + \frac{1}{r\sin\theta}\frac{\partial\mu}{\partial\phi}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial\theta} - \frac{u_\phi}{r}\cot\theta\right), \\
E_\phi &= \frac{\Delta}{r\sin\theta}\frac{\partial\lambda}{\partial\phi} + \frac{\partial\mu}{\partial r}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right) \\
&\quad + \frac{1}{r}\frac{\partial\mu}{\partial\theta}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial\theta} - \frac{u_\phi}{r}\cot\theta\right) \\
&\quad + \frac{2}{r\sin\theta}\cdot\frac{\partial\mu}{\partial\phi}\left(\frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right).
\end{aligned}$$

Since in the subsequent applications of equations (18) it is to be assumed that λ and μ are functions of r only, the expressions (19) are much simplified.*

* Considerable algebraic work is involved in the deduction of (19), but it is merely the routine work of transformation of coördinates and is therefore omitted. Although the application in the present paper is limited to the case in which λ and μ are independent of θ and ϕ, it seems desirable to record the general expressions.

6. **The boundary conditions.** The condition which will be assumed at the boundary surface is that the increment of stress on this surface is zero.* Making the same assumptions as in equations (3), but using polar coördinates, the increments of normal and tangential stress on a surface initially perpendicular to the radius vector are

$$
\begin{aligned}
\delta_{\widehat{rr}} &= \lambda\Delta + 2\mu \frac{\partial u_r}{\partial r}, \\
(20) \qquad \delta_{\widehat{r\theta}} &= \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right), \\
\delta_{\widehat{r\phi}} &= \mu\left(\frac{\partial u_\phi}{r\partial} - \frac{u_\phi}{r} + \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi}\right);
\end{aligned}
$$

and these are to vanish when $r = a$.

7. **Case in which the disturbing potential is a spherical harmonic function of the coördinates and a simple harmonic function of the time.** We now assume that the disturbing potential is proportional to $S_i r^i$, where S_i is a spherical surface harmonic of order i and a simple harmonic function of the time. In this case equations (18) may be reduced to simultaneous ordinary differential equations by the following method.†

Assume

$$
(21) \qquad u_r = uS_i, \qquad u_\theta = v\frac{\partial S_i}{\partial \theta}, \qquad u_\phi = \frac{v}{\sin\theta}\frac{\partial S_i}{\partial \phi},
$$

in which u, v are functions of r only. Substituting in the known formulas for Δ, ϖ_r, ϖ_θ, ϖ_ϕ, making use of the partial differential equation

$$
(22) \qquad \frac{1}{\sin^2\theta}\frac{\partial^2 S_i}{\partial \phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S_i}{\partial\theta}\right) + i(i+1)S_i = 0,
$$

and introducing the notation

$$
(23) \qquad y = \frac{1}{r^2}\frac{d(r^2 u)}{dr} - i(i+1)\frac{v}{r}, \qquad z = \frac{1}{r}\left(\frac{d(rv)}{dr} - u\right),
$$

we find

$$
(24) \quad \Delta = yS_i, \qquad 2\varpi_r = 0, \qquad 2\varpi_\theta = -\frac{z}{\sin\theta}\frac{\partial S_i}{\partial\phi}, \qquad 2\varpi_\phi = z\frac{\partial S_i}{\partial\theta};
$$

so that the first members of (18), exclusive of the terms E_r, E_θ, E_ϕ, take

* The amendment to the physical theory, referred to above, affects the boundary conditions as well as the differential equations. The increment of stress is zero, not for the fixed surface $r = a$, but for the material surface initially at $r = a$.

† This method was used in the writer's previous papers dealing with less general cases of the sphere problem. Some of the details given in those papers are here omitted. See these Transactions, vol. 11 (1910), p. 238 et seq.; also p. 495 et seq.

the forms

$$(25)\quad \left[(\lambda+2\mu)\frac{dy}{dr}+i(i+1)\mu\frac{z}{r}\right]S_i, \qquad \left[(\lambda+2\mu)\frac{y}{r}+\frac{\mu}{r}\frac{d(rz)}{dr}\right]\frac{\partial S_i}{\partial\theta},$$
$$\left[(\lambda+2\mu)\frac{y}{r}+\frac{\mu}{r}\frac{d(rz)}{dr}\right]\frac{1}{\sin\theta}\frac{\partial S_i}{\partial\phi};$$

while from (19),

$$(26)\quad E_r=\left(\frac{d\lambda}{dr}y+2\frac{d\mu}{dr}\frac{du}{dr}\right)S_i, \qquad E_\theta=\frac{d\mu}{dr}\left(\frac{u}{r}+\frac{dv}{dr}-\frac{v}{r}\right)\frac{\partial S_i}{\partial\theta},$$
$$E_\phi=\frac{d\mu}{dr}\left(\frac{u}{r}+\frac{dv}{dr}-\frac{v}{r}\right)\frac{1}{\sin\theta}\frac{\partial S_i}{\partial\phi}.$$

Again, since S_i is a simple harmonic function of t, we may put

$$(27)\quad \frac{\partial^2 S_i}{\partial t^2}=-p^2 S_i,$$

in which p is a constant. Hence the acceleration components in (18) become

$$(28)\quad \frac{\partial^2 u_r}{\partial t^2}=-p^2 uS_i, \qquad \frac{\partial^2 u_\theta}{\partial t^2}=-p^2 v\frac{\partial S_i}{\partial\theta}, \qquad \frac{\partial^2 u_\phi}{\partial t^2}=-\frac{p^2 v}{\sin\theta}\frac{\partial S_i}{\partial\phi}.$$

It remains to express the values of V and V' for substitution in the second members of (18).

Let the potential of the disturbing forces be

$$(29)\quad W=\frac{cg}{2a^{i-1}}r^i S_i,$$

c being a constant which measures the intensity of the disturbing force, and g the surface value of the gravitational attraction per unit mass.

If ρ_m is the mean density of the sphere and a the radius,

$$(30)\quad V=\frac{3g}{\rho_m a}\left[\frac{1}{r}\int_0^r \rho r^2\,dr+\int_r^a \rho r dr\right].$$

The potential V' is made up of W and the gravitation potential of the strained body; hence if U denotes the increment of the gravitation potential caused by the strain,

$$(31)\quad V'-V=W+U.$$

The value of U may be expressed as the sum of two parts, one of which is the potential of a distribution of matter of density

$$-\rho\Delta-u_r\frac{d\rho}{dr}$$

throughout the sphere, the other that of a surface layer of thickness equal to the surface value of u_r. Since, by virtue of (21) and (24),

$$-\rho\Delta - u_r\frac{d\rho}{dr} = -\left(\rho y + u\frac{d\rho}{dr}\right)S_i, \tag{32}$$

the former of the two parts of U is

$$-\frac{4\pi\gamma S_i}{2i+1}\left[\frac{1}{r^{i+1}}\int_0^r\left(\rho y + u\frac{d\rho}{dr}\right)r^{i+2}\,dr + r^i\int_r^a\left(\rho y + u\frac{d\rho}{dr}\right)\frac{dr}{r^{i-1}}\right], \tag{33}$$

while the latter is

$$\frac{4\pi\gamma S_i}{2i+1}\rho_1 u_1 a\left(\frac{r}{a}\right)^i, \tag{34}$$

in which the suffix ($_1$) refers to surface values. Combining (33) and (34) and integrating by parts the terms containing $d\rho/dr$, we find

$$\begin{aligned} U = -\frac{3gS_i}{(2i+1)\rho_m a}\left[\frac{1}{r^{i+1}}\int_0^r \rho\left(yr^{i+2} - \frac{d}{dr}(ur^{i+2})\right)dr\right. \\ \left. + r^i\int_r^a \rho\left(\frac{y}{r^{i-1}} - \frac{d}{dr}\left(\frac{u}{r^{i-1}}\right)\right)dr\right], \end{aligned} \tag{35}$$

in which $4\pi\gamma$ has been replaced by $3g/\rho_m a$. The value of $V' - V$ is obtained by combining (29) and (35), in accordance with (31).

From (21), (24), and (30),

$$u_r\frac{dV}{dr} = -\frac{3guS_i}{\rho_m ar^2}\int_0^r \rho r^2\,dr, \qquad \Delta\frac{dV}{dr} = -\frac{3gyS_i}{\rho_m ar^2}\int_0^r \rho r^2\,dr. \tag{36}$$

The final result of the substitution of (21) in (18) is obtained by combining the foregoing partial results. The first members of the resulting equations are made up of (25) and (26), while the second members are obtained from (28), (29), (35), and (36). After cancellation of the factors S_i, $\partial S_i/\partial\theta$, and $(1/\sin\theta)(\partial S_i/\partial\phi)$, the three equations are independent of θ, ϕ, and t, the second and third equations being in fact identical. The solution of the system (18) is thus reduced to that of two simultaneous ordinary differential equations with r as independent variable and u, v as the functions to be determined. These two equations, after multiplication by r, may be written as follows:

$$\begin{aligned} (\lambda + 2\mu)r\frac{dy}{dr} + i(i+1)\mu z + r\frac{d\lambda}{dr}y + 2r\frac{d\mu}{dr}\frac{du}{dr} \\ = \frac{g\rho a}{\rho_m}\left(r\frac{dR}{dr} - \frac{3y}{a^2 r}\int_0^r \rho r^2\,dr\right) - \rho p^2 ru, \end{aligned} \tag{37}$$

$$(\lambda + 2\mu)y + \mu\frac{d(rz)}{dr} + r\frac{d\mu}{dr}\left(\frac{u}{r} + \frac{dv}{dr} - \frac{v}{r}\right) = \frac{g\rho a}{\rho_m}R - \rho p^2 rv\,;$$

$$R = \frac{\rho_m}{gaS_i}\left(V - V' - u_r \frac{dV}{dr}\right)$$

$$(38) \qquad = \frac{3}{(2i+1)a^2}\left[\frac{1}{r^{i+1}}\int_0^r \rho\left(yr^{i+2} - \frac{d}{dr}(ur^{i+2})\right)dr \right.$$

$$\left. + r^i \int_r^a \rho\left(\frac{y}{r^{i-1}} - \frac{d}{dr}\left(\frac{u}{r^{i-1}}\right)\right)dr\right] + \frac{3u}{a^2 r^2}\int_0^r \rho r^2\, dr - \frac{c\rho_m}{2}\left(\frac{r}{a}\right)^i.$$

The substitution of (21) in (20) reduces the boundary conditions to the two equations

$$(39) \qquad \lambda y + 2\mu \frac{du}{dr} = 0, \qquad \mu\left(\frac{dv}{dr} - \frac{v}{r} + \frac{u}{r}\right) = 0,$$

to be satisfied when $r = a$.

It will be shown that the solution of (37) subject to the conditions (39) may be obtained in the form of infinite convergent series in a comprehensive class of cases in which ρ, λ, and μ are variable. The solution will first be given on the assumption that λ and μ are constants while ρ is expressible as a rational integral function of r. It will then be shown that the effect of assuming λ and μ also to be rational integral functions of r is merely to add terms to the main equations employed in the solution, and that when these functions are restricted in a certain way the series expressing u and v are still convergent.

III. Solution of Equations when the Density Function is Rational and Integral and the Elastic Moduli are Constants

1. **Solution when both elastic moduli are finite.** Assuming λ and μ to be constants, let

$$(40) \qquad \rho = \rho_0(1 + k_1 x + k_2 x^2 + \cdots),$$

in which $\rho_0, k_1, k_2, \cdots$ are constants, and

$$(41) \qquad x = \frac{r}{a}.$$

Hereafter x will replace r as independent variable, and u, v will be replaced by e, α, defined by the equations*

$$(42) \qquad e = \frac{u}{r}, \qquad \alpha = \frac{v}{r}.$$

Also, since $p^2 a$ is of the same dimensions as acceleration, let

$$(43) \qquad p^2 a = ng.$$

* It will later appear that, in the important case $i = 2$, e and α have simple physical meanings. It is seen that x, e, and α (as well as y and z already defined) are abstract numbers.

Equations (37) and (38) now take the following forms:

$$
(44)\qquad
\begin{aligned}
&(\lambda + 2\mu)\, x\frac{dy}{dx} + i(i+1)\,\mu z = \frac{g\rho a}{\rho_m}\left(x\frac{dR}{dx} - \frac{3y}{x}\int_0^x \rho x^2\, dx - n\rho_m\, ex^2 \right),\\
&(\lambda + 2\mu)\,\dot{y} + \mu\frac{d(xz)}{dx} = \frac{g\rho a}{\rho_m}\,(R - n\rho_m\, \alpha x^2),
\end{aligned}
$$

$$
(45)\qquad
\begin{aligned}
R = \frac{3}{2i+1}\Bigg[&\frac{1}{x^{i+1}}\int_0^x \rho\left(yx^{i+2} - \frac{d}{dx}(ex^{i+3}) \right) dx\\
&+ x^i \int_x^1 \rho\left(\frac{y}{x^{i-1}} - \frac{d}{dx}\left(\frac{e}{x^{i-2}} \right) \right) dx \Bigg] + \frac{3e}{x}\int_0^x \rho x^2\, dx - \frac{c\rho_m}{2}\, x^i;
\end{aligned}
$$

while the boundary conditions (39) become

$$
(46)\qquad \lambda y + 2\mu\frac{d(ex)}{dx} = 0, \qquad \mu\left(x\frac{d\alpha}{dx} + e \right) = 0, \qquad (\text{when } x = 1).
$$

If we now assume

$$
(47)\qquad
\begin{aligned}
y &= \sum_{m=i}^{\infty} C_m\, x^m, &\qquad z &= \sum_{m=i}^{\infty} D_m\, x^m,\\
e &= \sum_{m=i-2}^{\infty} A_m\, x^m, &\qquad \alpha &= \sum_{m=i-2}^{\infty} B_m\, x^m,
\end{aligned}
$$

it is found that the coefficients C_m, D_m, A_m, B_m may be so determined as to satisfy identically both (44) and (46). The solution may proceed as follows.

Notice first that A_m and B_m may be expressed in terms of C_m and D_m by means of (23), which may now be written

$$
(48)\qquad y = \frac{1}{x^2}\frac{d(ex^3)}{dx} - i(i+1)\,\alpha, \qquad z = \frac{1}{x}\frac{d(\alpha x^2)}{dx} - e.
$$

These give

$$
(49)\qquad
\begin{aligned}
C_m &= (m+3)\,A_m - i(i+1)\,B_m,\\
D_m &= -\,A_m + (m+2)\,B_m,
\end{aligned}
$$

which must hold for all values of m appearing in the assumed series (47). Since the series for y and z contain no terms of degree* $i-1$ or $i-2$, equations (49) require that

$$
(50)\qquad A_{i-2} - iB_{i-2} = 0,
$$

$$
(51)\qquad A_{i-1} = B_{i-1} = 0;
$$

while for all other values of m,

* It will later appear that in the case of fluid equilibrium ($\mu = 0$ and $n = 0$), terms of degree $i-1$ must be introduced unless $k_1 = 0$. This exception has no effect on the general method of solution.

$$
(52) \qquad \begin{aligned}
A_m &= \frac{(m+2)C_m + i(i+1)D_m}{(m+2)(m+3) - i(i+1)}, \\
B_m &= \frac{C_m + (m+3)D_m}{(m+2)(m+3) - i(i+1)}.
\end{aligned}
$$

Introducing now the notation

$$
(53) \qquad k' = \rho_0/\rho_m, \qquad b = g\rho_m a/\mu,
$$

we proceed to substitute the assumed values of e, α, y, z in (44) and (45).

The expression for R takes the form

$$
(54) \qquad R = \rho_0 \sum_{m=i}^{\infty} L_m x^m,
$$

in which

$$
(55) \qquad \begin{aligned}
L_i = \frac{3}{2i+1} \sum_{m=i}^{\infty} (C_m - (m-i+2)A_m)\left(\frac{1}{m-i+2} + \frac{k_1}{m-i+3}\right. \\
\left. + \frac{k_2}{m-i+4} + \cdots\right) + \frac{2i-2}{2i+1} A_{i-2} - \frac{c}{2k'},
\end{aligned}
$$

while for $m = i+1, i+2, \cdots$

$$
(56) \qquad \begin{aligned}
L_m = -\frac{3}{(m-i)(m+i+1)} (C_{m-2} + k_1 C_{m-3} + k_2 C_{m-4} + \cdots \\
+ k_1 A_{m-3} + 2k_2 A_{m-4} + 3k_3 A_{m-5} + \cdots) \\
+ 3(\tfrac{1}{3}A_{m-2} + \tfrac{1}{4}k_1 A_{m-3} + \tfrac{1}{5}k_2 A_{m-4} + \cdots).
\end{aligned}
$$

Equations (44) become

$$
(57) \qquad \begin{aligned}
&\sum\left[\left(\frac{\lambda}{\mu} + 2\right) mC_m + i(i+1)D_m\right] x^m \\
&\quad = bk'^2(1 + k_1 x + k_2 x^2 + \cdots)\left[\sum mL_m x^m - 3\sum C_m (\tfrac{1}{3}x^{m+2}\right. \\
&\qquad\qquad \left. + \tfrac{1}{4}k_1 x^{m+3} + \cdots) - \frac{n}{k'}\sum A_m x^{m+2}\right], \\
&\sum\left[\left(\frac{\lambda}{\mu} + 2\right) C_m + (m+1)D_m\right] x^m \\
&\quad = bk'^2(1 + k_1 x + k_2 x^2 + \cdots)\left[\sum L_m x^m - \frac{n}{k'}\sum B_m x^{m+2}\right];
\end{aligned}
$$

the summations extending to all values of m occurring in the assumed expansions (47). These equations are to be satisfied identically.

The right-hand members contain no term of degree lower than i. The same is true of the left-hand members if (50) and (51) are satisfied. Hence to satisfy (57) identically it is necessary and sufficient to satisfy the following

equations for $m = i, i+1, i+2, \cdots$:

$$\left(\frac{\lambda}{\mu}+2\right) mC_m + i(i+1)D_m = bk'^2\Big[mL_m + k_1(m-1)L_{m-1}$$

$$+ k_2(m-2)L_{m-2} + \cdots - (C_{m-2} + k_1' C_{m-3} + k_2' C_{m-4} + \cdots)$$

$$\text{(58)} \qquad - \frac{n}{k'}(A_{m-2} + k_1 A_{m-3} + k_2 A_{m-4} + \cdots)\Big],$$

$$\left(\frac{\lambda}{\mu}+2\right) C_m + (m+1)D_m = bk'^2\Big[L_m + k_1 L_{m-1} + k_2 L_{m-2} + \cdots$$

$$- \frac{n}{k'}(B_{m-2} + k_1 B_{m-3} + k_2 B_{m-4} + \cdots)\Big],$$

in which $k_1', k_2', \cdots$ are easily-determined functions of $k_1, k_2, \cdots$.

For $m = i$ the two equations are identical:

$$\text{(59)} \qquad \left(\frac{\lambda}{\mu}+2\right) C_i + (i+1)D_i = bk'^2\left(L_i - \frac{n}{ik'} A_{i-2}\right);$$

while for $m = i+1, i+2, \cdots$, they may be solved for C_m and D_m, giving

$$(m-i)(m+i+1)D_m = bk'^2\Big[k_1 L_{m-1} + 2k_2 L_{m-2} + 3k_3 L_{m-3}$$

$$+ \cdots + C_{m-2} + k_1' C_{m-3} + k_2' C_{m-4} + \cdots + \frac{n}{k'}(A_{m-2} - mB_{m-2}$$

$$\text{(60)} \qquad + k_1(A_{m-3} - mB_{m-3}) + k_2(A_{m-4} - mB_{m-4}) + \cdots)\Big],$$

$$\left(\frac{\lambda}{\mu}+2\right) C_m = -(m+1)D_m + bk'^2\Big[L_m + k_1 L_{m-1} + k_2 L_{m-2}$$

$$+ \cdots - \frac{n}{k'}(B_{m-2} + k_1 B_{m-3} + k_2 B_{m-4} + \cdots)\Big].$$

These equations serve to determine any pair of coefficients C_m, D_m (except C_i, D_i) in terms of coefficients of lower order. Remembering the relations (50), (51), (52), (55), and (56), it is seen that every C_m and D_m may, by successive applications of (60), be ultimately expressed in terms of the three coefficients C_i, D_i, A_{i-2}. When this has been accomplished the only non-vanishing terms in (57) will be those of degree i; and these will vanish when (59) is satisfied. Equation (59) in fact becomes one of three linear equations for determining C_i, D_i, and A_{i-2}; the other two being obtained from the surface conditions (46). These three equations may be written as follows:

$$\left(\frac{\lambda}{\mu}+2\right)C_i+(i+1)D_i=bk'^2\left[\frac{3}{2i+1}\sum_{m=i}^{\infty}(C_m-(m-i+2)A_m)\right.$$
$$\times\left(\frac{1}{m-i+2}+\frac{k_1}{m-i+3}+\frac{k_2}{m-i+4}+\cdots\right)$$
$$\left.+\frac{2i-2}{2i+1}A_{i-2}-\frac{c}{2k'}-\frac{n}{ik'}A_{i-2}\right],\tag{61}$$
$$\lambda\sum_{m=i}^{\infty}C_m+2\mu\sum_{m=i-2}^{\infty}(m+1)A_m=0,\qquad \mu\sum_{m=i-2}^{\infty}(mB_m+A_m)=0.$$

The foregoing solution is valid for any constant values of λ and μ which are compatible with the assumption that the strain is "small," i.e., that only the first powers of the displacements are of appreciable magnitude. The statement of the solution however requires modification in the limiting cases of incompressibility ($\lambda=\infty$) and fluidity ($\mu=0$).

2. **Case of incompressibility.** If $\lambda=\infty$ the cubical expansion Δ must be 0, hence in applying the general solution to this case we must put $y=0$ while λy remains finite. Putting

$$\frac{\lambda}{\mu}y=y',\tag{62}$$

equations (44) and (45) become

$$x\frac{dy'}{dx}+i(i+1)z=\frac{g\rho a}{\mu\rho_m}\left(x\frac{dR}{dx}-n\rho_m ex^2\right),$$
$$y'+\frac{d(xz)}{dx}=\frac{g\rho a}{\mu\rho_m}(R-n\rho_m\alpha x^2),\tag{63}$$

$$R=-\frac{3}{2i+1}\left[\frac{1}{x^{i+1}}\int_0^x\rho\frac{d}{dx}(ex^{i+3})\,dx+x^i\int_x^1\rho\frac{d}{dx}\left(\frac{e}{x^{i-2}}\right)dx\right]$$
$$+\frac{3e}{x}\int_0^x\rho x^2\,dx-\frac{1}{2}c\rho_m x^i.\tag{64}$$

Without restricting the law of density, a linear differential equation for determining e may be obtained by eliminating y' between the two equations (63), substituting for z and α their values in terms of e as given by (48) with $y=0$, and substituting for R its value given by (64). By so operating as to eliminate the integrals containing e, there results a linear differential equation of the sixth order whose coefficients depend upon ρ and its derivatives.* This equation will not be given; it may be noted, however, that the elimination of the integrals containing e is accomplished by operating upon (64) as indicated

* In the case $n=0$ this equation becomes identical with that obtained by Herglotz in the paper cited above. His method of procedure is, however, quite different.

in the following equation:

$$
(65)\qquad \begin{aligned} & x^2\frac{d^2 R}{dx^2} + 2x\frac{dR}{dx} - i(i+1)R \\ & \qquad = 3\left(x\frac{d^2 e}{dx^2} - i(i+1)\frac{e}{x}\right)\int_0^x \rho x^2\,dx + 6\rho x^2\left(x\frac{de}{dx} + e\right). \end{aligned}
$$

When the density function is rational and integral, the above method of solution by infinite series applies with slight modification to the case of incompressibility, the equations expressing the solution being in fact much simplified.

Assume

$$
(66)\qquad y' = \sum_{m=i}^{\infty} K_m x^m,
$$

and equate to 0 every term in the equations which contains C_m unless multiplied by λ, but

$$
(67)\qquad \frac{\lambda}{\mu} C_m = K_m.
$$

The main equations in the general solution then take the following forms.

$$
(68)\qquad \begin{aligned} L_i = & -\frac{3}{2i+1}\sum_{m=i}^{\infty}(m-1+2)A_m\left(\frac{1}{m-i+2} + \frac{k_1}{m-i+3}\right. \\ & \left. + \frac{k_2}{m-i+4} + \cdots\right) + \frac{2i-2}{2i+1}A_{i-2} - \frac{v}{2k'}, \end{aligned}
$$

$$
(69)\qquad \begin{aligned} L_m = & \; 3\left(\tfrac{1}{3}A_{m-2} + \tfrac{1}{4}k_1 A_{m-3} + \tfrac{1}{5}k_2 A_{m-4} + \cdots\right) \\ & - \frac{3}{(m-i)(m+i+1)}(k_1 A_{m-3} + 2k_2 A_{m-4} + 3k_3 A_{m-5} + \cdots), \end{aligned}
$$

$$
(70)\qquad \begin{aligned} mK_m + i(i+1)D_m = & \; bk'^2\Big[mL_m + k_1(m-1)L_{m-1} + k_2(m-2)L_{m-2} \\ & + \cdots - \frac{n}{k'}(A_{m-2} + k_1 A_{m-3} + k_2 A_{m-4} + \cdots)\Big], \\ K_m + (m+1)D_m = & \; bk'^2\Big[L_m + k_1 L_{m-1} + k_2 L_{m-2} + \cdots \\ & - \frac{n}{k'}(B_{m-2} + k_1 B_{m-3} + k_2 B_{m-4} + \cdots)\Big], \end{aligned}
$$

$$
(71)\qquad K_i + (i+1)D_i = bk'^2\left(L_i - \frac{n}{ik'}A_{i-2}\right),
$$

$$
(72)\quad
\begin{aligned}
(m-i)(m+i+1)D_m &= bk'^2\Big[k_1 L_{m-1} + 2k_2 L_{m-2} + 3k_3 L_{m-3} \\
&\qquad + \cdots + \frac{n}{k'}(A_{m-2} - mB_{m-2} + k_1(A_{m-3} - mB_{m-3}) + \cdots)\ , \\
K_m &= -(m+1)D_m + bk'^2\Big[L_m + k_1 L_{m-1} + k_2 L_{m-2} + k_3 L_{m-3} \\
&\qquad + \cdots - \frac{n}{k'}(B_{m-2} + k_1 B_{m-3} + \cdots)\Big].
\end{aligned}
$$

These correspond respectively to equations (55), (56), (58), (59), and (60).

Proceeding as before, equations (72) serve to determine every K_m and D_m in terms of K_i, D_i, and A_{i-2}. The three equations for determining these (corresponding to (61)) are

$$
(73)\quad
\begin{aligned}
&K_i + (i+1)D_i = bk'^2\Big[-\frac{3}{2i+1}\sum_{m=i}^{\infty}(m-i+2)\Big(\frac{1}{m-i+2} + \\
&\frac{k_1}{m-i+3} + \frac{k_2}{m-i+4} + \cdots\Big)A_m + \Big(\frac{2i-2}{2i+1} - \frac{n}{ik'}\Big)A_{i-2} - \frac{c}{2k'}\Big], \\
&\sum_{m=i}^{\infty} K_m + 2\sum_{m=i-2}^{\infty}(m+1)A_m = 0, \qquad \mu\sum_{m=i-2}^{\infty}(mB_m + A_m) = 0.
\end{aligned}
$$

3. **Case of fluidity or zero rigidity.** If $\mu = 0$, both λ/μ and b become infinite, but their ratio is finite. Hence we multiply equations (57) by μ/λ and introduce the notation

$$
(74)\qquad b' = \frac{\mu b}{\lambda} = \frac{g\rho_m a}{\lambda}.
$$

The first members of these equations become

$$
(75)\qquad \sum mC_m x^m \quad \text{and} \quad \sum C_m x^m,
$$

while the second members are unchanged except by the substitution of b' for b. Equations (58) therefore become

$$
(76)\quad
\begin{aligned}
mC_m &= b'k'^2\Big[mL_m + k_1(m-1)L_{m-1} + k_2(m-2)L_{m-2} + \cdots \\
&\qquad - (C_{m-2} + k_1' C_{m-3} + k_2' C_{m-4} + \cdots) \\
&\qquad - \frac{n}{k'}(A_{m-2} + k_1 A_{m-3} + k_2 A_{m-4} + \cdots)\Big], \\
C_m &= b'k'^2\Big[L_m + k_1 L_{m-1} + k_2 L_{m-2} + \cdots \\
&\qquad - \frac{n}{k'}(B_{m-2} + k_1 B_{m-3} + k_2 B_{m-4} + \cdots)\Big],
\end{aligned}
$$

which must be satisfied for $m = i, i+1, \cdots$. As before, these equations are identical when $m = i$, giving

$$C_i = b' k'^2 \left(L_i - \frac{nA_{i-2}}{ik'} \right) \tag{77}$$

corresponding to (59); while for other values of m they must be solved simultaneously. The simplest procedure seems to be to eliminate C_m as before, giving an equation similar to the first of (60) but with the first member zero. If the solution is carried out with successive values $m = i+1, i+2, \cdots$, having due regard for equations (50), (51), and (52), every coefficient may be determined in terms of those of lower order, and ultimately in terms of those of lowest order. It is found, however, that when (76) are satisfied for $m = i+1, i+2, \cdots$, only two coefficients remain arbitrary; all coefficients may in fact be determined in terms of D_i and A_{i-2} so as to reduce to zero every term in the equations corresponding to (57) except those of degree i. In order to cause these to vanish, and also to satisfy the boundary conditions, it is necessary and sufficient to so determine D_i and A_{i-2} as to satisfy the equations corresponding to (61). Since $\mu = 0$ the third of these is satisfied independently of D_i and A_{i-2}, while the first and second take the forms

$$\begin{aligned} C_i = b' k'^2 \Bigg[\frac{3}{2i+1} \sum_{m=i}^{\infty} (C_m - (m - i + 2) A_m) \\ \times \left(\frac{1}{m-i+2} + \frac{k_1}{m-i+3} + \frac{k_2}{m-i+4} + \cdots \right) \\ + \frac{2i-2}{2i+1} A_{i-2} - \frac{c}{2k'} - \frac{n}{ik'} A_{i-2} \Bigg], \\ \sum_{m=i}^{\infty} C_m = 0. \end{aligned} \tag{78}$$

Special case of fluid equilibrium. If $n = 0$ and $k_1 \neq 0$, the above solution fails unless terms of degree $i - 1$ are introduced in the four series (47). This case is best treated independently, especially as it is possible thereby to show that the above general theory includes as a special case the ordinary theory of the equilibrium of a fluid sphere of variable density.

It is easy to show that, if $n = 0$, equations (44) cannot be satisfied with an arbitrary law of density except by making

$$y = 0, \qquad R = 0. \tag{79}$$

Substituting the value of R given by (45) with $y = 0$ in the equation

$$\frac{d}{dx}\left(x^2 \frac{dR}{dx} \right) - i(i+1)R = 0, \tag{80}$$

and noting equation (65), there results

$$\text{(81)} \qquad \frac{i(i+1)e - x^2 \dfrac{d^2 e}{dx^2}}{e + x\dfrac{de}{dx}} = \frac{2\rho x^3}{\displaystyle\int_0^x \rho x^2\,dx},$$

a well-known equation in the theory of equilibrium of a fluid sphere of variable density.

The solution by infinite series when the density function is rational and integral is easily carried out if terms of degree $i - 1$ are introduced in the series.* The analysis in fact simplifies greatly from the fact that

$$\text{(82)} \qquad C_m = 0 \quad \text{and} \quad L_m = 0$$

for every value of m.

IV. Solution of equations for cases of variable elastic moduli

1. **The differential equations and surface conditions.** The differential equations (37) and the surface conditions (39) cover the general case in which λ and μ are any functions of r. When λ and μ are constants (37) reduce to (44). Restoring the terms involving derivatives of λ and μ, there must be added to the first members of (44) the terms

$$\text{(83)} \qquad xy\frac{d\lambda}{dx} + 2x\frac{d(ex)}{dx}\frac{d\mu}{dx}, \qquad x\left(e + x\frac{d\alpha}{dx}\right)\frac{d\mu}{dx}.$$

The surface conditions (46) are unchanged, since they do not involve derivatives of λ or μ.

2. **Elastic moduli assumed to be rational integral functions of r.** We now assume that λ and μ are expressible by polynomials in x:

$$\text{(84)} \qquad \lambda = \lambda_0(1 + s_1 x + s_2 x^2 + \cdots), \qquad \mu = \mu_0(1 + t_1 x + t_2 x^2 + \cdots),$$

and proceed to examine the effect of the terms (83) upon the solution by infinite series. The density is still assumed to be given by (40).

The substitution of the assumed series (47) for y, z, e, α reduces the differential equations to two equations similar to (57), but with the addition of terms derived from (83) and (84). If we let

$$\text{(85)} \qquad b = g\rho_m a/\mu_0$$

(which includes the definition of b given by (53)), the second members of the new equations will be identical with those of (57), while the first members will be the following:

* This is unnecessary if $k_1 = 0$ in the density function.

$$
(86)\quad
\begin{aligned}
&\sum\Big\{\Big(\frac{\lambda_0}{\mu_0}+2\Big)mC_m+i(i+1)D_m+\Big[\frac{\lambda_0}{\mu_0}ms_1+2(m-1)t_1\Big]C_{m-1}\\
&\quad+\Big[\frac{\lambda_0}{\mu_0}ms_2+2(m-2)t_2\Big]C_{m-2}+\Big[\frac{\lambda_0}{\mu_0}ms_3+2(m-3)t_3\Big]C_{m-3}+\cdots\\
&\quad+i(i+1)(t_1D_{m-1}+t_2D_{m-2}+\cdots)\\
&\quad+2[mt_1A_{m-1}+2(m-1)t_2A_{m-2}+3(m-2)t_3A_{m-3}+\cdots]\Big\}x^m,\\
&\sum\Big\{\Big(\frac{\lambda_0}{\mu_0}+2\Big)C_m+(m+1)D_m+\Big(\frac{\lambda_0}{\mu_0}s_1+2t_1\Big)C_{m-1}\\
&\quad+\Big(\frac{\lambda_0}{\mu_0}s_2+2t_2\Big)C_{m-2}+\Big(\frac{\lambda_0}{\mu_0}s_3+2t_3\Big)C_{m-3}+\cdots\\
&\quad+mt_1D_{m-1}+(m-1)t_2D_{m-2}+(m-2)t_3D_{m-3}+\cdots\\
&\quad+t_1[A_{m-1}+(m-1)B_{m-1}]+2t_2[A_{m-2}+(m-2)B_{m-2}]\\
&\quad+3t_3[A_{m-3}+(m-3)B_{m-3}]+\cdots\Big\}x^m.
\end{aligned}
$$

These expressions contain no terms of degree lower than i, except terms containing t_1 as a factor. Unless $t_1 = 0$ the series for y, z, e, α must contain terms of degree $i-1$. No terms of degree less than $i+1$ are thereby introduced into the second members of the equations corresponding to (57), while the terms of lowest degree in the first members are

$$
(87)\quad
\begin{aligned}
&\Big[\Big(\frac{\lambda_0}{\mu_0}+2\Big)(i-1)C_{i-1}+i(i+1)D_{i-1}+2(i-1)t_1A_{i-2}\Big]x^{i-1},\\
&\Big[\Big(\frac{\lambda_0}{\mu_0}+2\Big)C_{i-1}+iD_{i-1}+t_1(A_{i-2}+(i-2)B_{i-2})\Big]x^{i-1};
\end{aligned}
$$

which must be made to vanish by determining C_{i-1} and D_{i-1} in terms of A_{i-2}, remembering (50).

In the subsequent analysis it will be assumed that $t_1 = s_1 = 0$,—a reasonable assumption in any geodynamical application, since it is natural to suppose that $d\lambda/dx$ and $d\mu/dx$ vanish at the center.

Using (86) instead of the first members of (57), and assuming the new equations to be satisfied identically, we obtain a pair of equations whose second members are identical with those of (58) and whose first members are the following:

$$
(88)\quad
\begin{aligned}
&\left(\frac{\lambda_0}{\mu_0}+2\right)mC_m + i(i+1)D_m + \left[\frac{\lambda_0}{\mu_0}ms_2 + 2(m-2)t_2\right]C_{m-2} \\
&\quad + \frac{\lambda_0}{\mu_0}\left[ms_3 + 2(m-3)t_3\right]C_{m-3} + \cdots \\
&\quad + i(i+1)(t_2 D_{m-2} + t_3 D_{m-3} + \cdots) \\
&\quad + 2[2(m-1)t_2 A_{m-2} + 3(m-2)t_3 A_{m-3} + \cdots], \\
&\left(\frac{\lambda_0}{\mu_0}+2\right)C_m + (m+1)D_m + \left(\frac{\lambda_0}{\mu_0}s_2 + 2t_2\right)C_{m-2} \\
&\quad + \left(\frac{\lambda_0}{\mu_0}s_3 + 2t_3\right)C_{m-3} + \cdots + (m-1)t_2 D_{m-2} \\
&\quad + (m-2)t_3 D_{m-3} + \cdots + 2t_2[A_{m-2} + (m-2)B_{m-2}] \\
&\quad + 3t_3[A_{m-3} + (m-3)B_{m-3}] + \cdots .
\end{aligned}
$$

For $m = i$ the two equations are identical, giving

$$
(89)\quad \left(\frac{\lambda_0}{\mu_0}+2\right)C_i + (i+1)D_i + \frac{4(i-1)}{i}t_2 A_{i-2} = bk'^2\left(L_i - \frac{n}{ik'}A_{i-2}\right).
$$

For $m > i$ they may be solved for C_m and D_m, giving these in terms of coefficients of lower order, and thus ultimately in terms of C_i, D_i, A_{i-2}. If the resulting series are convergent, the solution may be completed as in the case of uniform elasticity already treated; i.e., by substituting the values of C_m and D_m (expressed as linear functions of C_i, D_i, A_{i-2}) in (89) and in the boundary equations (46), thus obtaining three linear equations for determining C_i, D_i, A_{i-2}. These equations, corresponding to (61), are the following (λ_1 and μ_1 being surface values of λ and μ):

$$
(90)\quad
\begin{aligned}
&\left(\frac{\lambda_0}{\mu_0}+2\right)C_i + (i+1)D_i + \frac{4(i-1)}{i}t_2 A_{i-2} \\
&\quad = bk'^2\left\{\frac{3}{2i+1}\sum_{m=i}^{\infty}[C_m - (m-i+2)A_m]\left(\frac{1}{m-i+2}\right.\right. \\
&\quad \left.\left. + \frac{k_1}{m-i+3} + \cdots\right) + \frac{2i-2}{2i+1}A_{i-2} - \frac{c}{2k'} - \frac{n}{ik'}A_{i-2}\right\}, \\
&\frac{\lambda_1}{\mu_1}\sum_{m=i}^{\infty}C_m + 2\sum_{m=i-2}^{\infty}(m+1)A_m = 0, \qquad \mu_1\sum_{m=i-2}^{\infty}(mB_m + A_m) = 0.
\end{aligned}
$$

3. **Convergency of series.** When the above process of solving for C_m and D_m is carried out, it is probable that the resulting series do not converge unless important restrictions are imposed upon the constants s_m, t_m. The values

of C_m and D_m, so far as these depend upon terms occurring in (88), are as follows:

$$\begin{aligned}
&(m-i)(m+i+1)D_m \\
&\quad = -t_2[4C_{m-2} + (m^2 - m - i^2 - i)D_{m-2} - 2(m-2)A_{m-2} \\
&\qquad\qquad + 2m(m-2)B_{m-2}] \\
&\qquad - t_3[6C_{m-3} + (m^2 - 2m - i^2 - i)D_{m-3} - 3(m-4)A_{m-3} \\
&\qquad\qquad + 3m(m-3)B_{m-3}] \\
&\qquad - t_4[8C_{m-4} + (m^2 - 3m - i^2 - i)D_{m-4} - 4(m-6)A_{m-4} \\
&\qquad\qquad + 4m(m-4)B_{m-4}] \\
&\qquad - \cdots\cdots,
\end{aligned}$$

$$\begin{aligned}
&(m-i)(m+i+1)\left(\frac{\lambda_0}{\mu_0}+2\right)C_m \\
&\quad = -(m-i)(m+i+1)\frac{\lambda_0}{\mu_0}(s_2 C_{m-2} + s_3 C_{m-3} + s_4 C_{m-4} + \cdots) \\
&\qquad - t_2[2\{(m+1)(m-2) - i(i+1)\}C_{m-2} + 2i(i+1)D_{m-2} \\
&\qquad\qquad + (4(m-1)(m+1) - 2i(i+1))A_{m-2} \\
&\qquad\qquad - 2i(i+1)(m-2)B_{m-2}] \\
&\qquad - t_3[2\{(m+1)(m-3) - i(i+1)\}C_{m-3} + 3i(i+1)D_{m-3} \\
&\qquad\qquad + (6(m-2)(m+1) - 3i(i+1))A_{m-3} \\
&\qquad\qquad - 3i(i+1)(m-3)B_{m-3}] \\
&\qquad - t_4[2\{(m+1)(m-4) - i(i+1)\}C_{m-4} + 4i(i+1)D_{m-4} \\
&\qquad\qquad + (8(m-3)(m+1) - 4i(i+1))A_{m-4} \\
&\qquad\qquad - 4i(i+1)(m-4)B_{m-4}] \\
&\qquad - \cdots\cdots,
\end{aligned} \tag{91}$$

So far as convergence is concerned, the governing terms are those of highest degree in m, after every A_n and B_n has been replaced by its value in terms of C_n and D_n by equations (52). After division by $(m-i)(m+i+1)$ these governing terms (of degree 0 in m) are as follows:

$$\begin{aligned}
D_m &= -t_2 D_{m-2} - t_3 D_{m-3} - t_4 D_{m-4} - \cdots, \\
C_m &= -\frac{\lambda_0 s_2 + 2\mu_0 t_2}{\lambda_0 + 2\mu_0} C_{m-2} - \frac{\lambda_0 s_3 + 2\mu_0 t_3}{\lambda_0 + 2\mu_0} C_{m-3} - \cdots.
\end{aligned} \tag{92}$$

Inspection of these terms shows that it is possible to insure the convergence of the series C_m and D_m by making all the governing terms zero except one in each equation and making the coefficients in the non-vanishing terms less than unity in absolute value; for in such case each series will ultimately approach coincidence with a convergent geometric series. The solution is thus practicable if λ and μ are expressed by binomial functions

$$\lambda = \lambda_0(1 + s_n x^n), \quad \mu = \mu_0(1 + t_n x^n), \tag{93}$$

in which n is a positive integer and s_n, t_n are less than unity in absolute value.

Even with this restriction the solution is of very considerable interest, since it supplies a very important generalization in the case of the geodynamical problem of the elastic yielding of the earth.

V. Particular solutions

1. **Numerical applications.** The general solution above given has been applied in the computation of a considerable number of numerical results. The data used in these computations have been chosen in approximate conformity with the known dimensions and properties of the earth, and the results fall into several series designed to show separately the effects of different assumptions regarding density, compressibility, and rigidity. Since the labor increases considerably with each added term in the density formula, the computations have been restricted to the case in which the formula is a binomial. Fortunately this simple formula is capable of expressing a fair approximation to the facts for the earth so far as known. It has already been pointed out that the above solution for variable elasticity is restricted to the case in which λ and μ are expressed by binomials.

While omitting numerical details, it seems desirable to show the form to which the main formulas have been reduced for the purpose of computation. These will be given in a form sufficiently inclusive to cover all the special cases for which computations have been made.

2. **Formulas for density and elastic moduli.** The simplified formulas assumed for ρ, λ, μ are as follows:

$$\rho = \rho_0(1 - kx^2), \tag{94}$$

$$\lambda = \lambda_0(1 - hx^2), \qquad \mu = \mu_0(1 - hx^2), \tag{95}$$

in which k and h are positive and less than 1. It will be observed that (95) fall under (93), and also that they make λ/μ constant.

The surface values are

$$\rho_1 = \rho_0(1 - k), \tag{96}$$

(97) $$\lambda_1 = \lambda_0(1 - h), \qquad \mu_1 = \mu_0(1 - h),$$

while the mean density is

(98) $$\rho_m = \rho_0(1 - \tfrac{3}{5}k),$$

so that

(99) $$k' = \frac{\rho_0}{\rho_m} = \frac{5}{5 - 3k}.$$

The numerical values to be assigned to the constants k, h, ρ_0, λ_0, μ_0 in order to represent as nearly as possible the facts for the earth will be considered later.

3. **Disturbing potential of second degree.** The particular solutions which will be considered all fall under the case $i = 2$, which covers the actual case of tidal and centrifugal forces. The lunar or solar tidal potential is in fact expressed by (29) with

(100) $$S_i = S_2 = \cos^2\theta' - \tfrac{1}{3},$$

in which θ' is the zenith distance of the disturbing body. Referring to (42) and (21) it is seen that, with this value of S_2, e denotes the ellipticity of a surface which in the unstrained body would be spherical and of radius r, while α denotes the angular displacement of a radius vector drawn to a particle for which $\theta' = 45°$.

In the static problem we may take $\theta' = \theta$.

4. **Differential equations and boundary conditions.** We now return to equations (37), (38), and (39), and note the form taken by their solution in the case represented by (96), (97), and (100). After the introduction of x as independent variable the differential equations are

(101) $$\begin{gathered}(\lambda + 2\mu)\,x\frac{dy}{dx} + 6\mu z + xy\frac{d\lambda}{dx} + 2x\frac{d(ex)}{dx}\frac{d\mu}{dx} \\ = \frac{g\rho a}{\rho_m}\left(x\frac{dR}{dx} - \frac{3y}{x}\int_0^x \rho x^2\,dx - n\rho_m e x^2\right), \\ (\lambda + 2\mu)y + \mu\frac{d(xz)}{dx} + x\left(e + x\frac{d\alpha}{dx}\right)\frac{d\mu}{dx} = \frac{g\rho a}{\rho_m}(R - n\rho_m \alpha x^2),\end{gathered}$$

in which

(102) $$R = \frac{3}{5}\left[\frac{1}{x^3}\int_0^x \rho\left(yx^4 - \frac{d(ex^5)}{dx}\right) + x^2\int_x^1 \rho\left(\frac{y}{x} - \frac{de}{dx}\right)dx\right] + \frac{3e}{x}\int_0^x \rho x^2\,dx - \frac{c\rho_m x^2}{2};$$

while the boundary conditions are

(103) $$\lambda y + 2\mu\frac{d(ex)}{dx} = 0, \qquad \mu\left(e + x\frac{d\alpha}{dx}\right) = 0, \qquad (\text{when } x = 1).$$

These are obtained from (44), (45), and (46) by putting $i = 2$ and adding the terms (83) to the first members of (44).

Equations (23), defining y and z, now become

$$y = \frac{1}{x^2}\frac{d(ex^3)}{dx} - 6\alpha, \qquad z = \frac{1}{x}\frac{d(\alpha x^2)}{dx} - e. \tag{104}$$

5. **Solution of equations.** In the present case only even powers of x are required in the assumed series (47), which become

$$\begin{aligned} y &= C_2 x^2 + C_4 x^4 + \cdots & z &= D_2 x^2 + D_4 x^4 + \cdots \\ e &= A_0 + A_2 x^2 + A_4 x^4 + \cdots & \alpha &= B_0 + B_2 x^2 + B_4 x^4 + \cdots. \end{aligned} \tag{105}$$

Because of (104) the coefficients in (105) are related as follows:

$$C_m = (m+3)A_m - 6B_m, \qquad D_m = -A_m + (m+2)B_m. \tag{106}$$

This requires that

$$B_0 = \tfrac{1}{2}A_0, \tag{107}$$

while for $m = 2, 4, \cdots$, A_m and B_m may be expressed in terms of C_m and D_m by the formulas

$$A_m = \frac{(m+2)C_m + 6D_m}{m(m+5)}, \qquad B_m = \frac{C_m + (m+3)D_m}{m(m+5)}, \tag{108}$$

which are equations (52) with $i = 2$.

The solution now proceeds as outlined in equations (54)–(61), but taking account of the terms which must be added when λ and μ are variable. These terms are obtained from the results given in Part IV, noting that all the constants s_m, t_m in (84) vanish except s_2, t_2, each of which is now replaced by $-h$.

The value of R becomes

$$R = \rho_0(L_2 x^2 + L_4 x^4 + \cdots), \tag{109}$$

in which

$$L_2 = \frac{3}{5}\sum_{m=2}^{\infty}(C_m - mA_m)\left(\frac{1}{m} - \frac{k}{m+2}\right) + \frac{2}{5}A_0 - \frac{c}{2k'}, \tag{110}$$

while for $m = 4, 6, \cdots$

$$L_m = -\frac{3}{(m-2)(m+3)}(C_{m-2} - kC_{m-4} - 2kA_{m-4}) + A_{m-2} - \frac{3}{5}kA_{m-4}. \tag{111}$$

These correspond to equations (54), (55), and (56).

The substitution of (105) in (101) gives the equations corresponding to (57); the second members are in fact identical with those of (57), while the first members are given by (86), proper substitutions being made for k_1, k_2,

$\cdots, s_1, s_2, \cdots, t_1, t_2, \cdots$, and b being defined by (85). These equations are

$$\sum\Big[\Big(\frac{\lambda_0}{\mu_0}+2\Big)mC_m+6D_m-\Big(m\frac{\lambda_0}{\mu_0}+2(m-2)\Big)hC_{m-2}$$
$$-6hD_{m-2}-4(m-1)hA_{m-2}\Big]x^m$$
$$=bk'^2(1-kx^2)\sum\Big[mL_m\,x^m-3C_m\Big(\frac{1}{3}x^{m+2}-\frac{1}{5}kx^{m+4}\Big)$$

(112)
$$-\frac{n}{k'}A_m\,x^{m+2}\Big],$$

$$\sum\Big[\Big(\frac{\lambda_0}{\mu_0}+2\Big)C_m+(m+1)D_m-\Big(\frac{\lambda_0}{\mu_0}+2\Big)hC_{m-2}$$
$$-(m-1)hD_{m-2}-2h(A_{m-2}+(m-2)B_{m-2})\Big]x^m$$
$$=bk'^2(1-kx^2)\sum\Big[L_m\,x^m-\frac{n}{k'}B_m\,x^{m+2}\Big],$$

the summations extending to all values of m which are consistent with (105). Both equations thus take the form of series of even powers of x, beginning with x^2. To satisfy them identically it is necessary to satisfy the following equations for $m = 2, 4, 6, \cdots$:

$$\Big(\frac{\lambda_0}{\mu_0}+2\Big)mC_m+6D_m-h\Big[\Big(\frac{\lambda_0}{\mu_0}m+2(m-2)\Big)C_{m-2}$$
$$+6D_{m-2}+4(m-1)A_{m-2}\Big]$$
$$=bk'^2\Big[mL_m-k(m-2)L_{m-2}-C_{m-2}+\frac{8}{5}kC_{m-4}-\frac{3}{5}k^2C_{m-6}$$

(113)
$$-\frac{n}{k'}(A_{m-2}-kA_{m-4})\Big],$$

$$\Big(\frac{\lambda_0}{\mu_0}+2\Big)C_m+(m+1)D_m-h\Big[\Big(\frac{\lambda_0}{\mu_0}+2\Big)C_{m-2}+(m-1)D_{m-2}$$
$$+2A_{m-2}+2(m-2)B_{m-2}\Big]$$
$$=bk'^2\Big[L_m-kL_{m-2}-\frac{n}{k'}(B_{m-2}-kB_{m-4})\Big].$$

For $m = 2$ the two equations are identical, giving

(114)
$$\Big(\frac{\lambda_0}{\mu_0}+2\Big)C_2+3D_2-2hA_0=bk'^2\Big(L_2-\frac{n}{2k'}A_0\Big);$$

while for $m = 4, 6, \cdots$, they may be solved for C_m and D_m, giving

$$D_m = \frac{bk'^2}{(m-2)(m+3)}\Big[-2kL_{m-2} + C_{m-2} - \frac{8}{5}kC_{m-4} + \frac{3}{5}k^2 C_{m-6} + \frac{n}{k'}\left(A_{m-2} - mB_{m-2} - k\left(A_{m-4} - mB_{m-4}\right)\right)\Big] + h\left(D_{m-2} + \frac{4C_{m-2} - 4D_{m-2}}{(m-2)(m+3)}\right),$$

$$(115)\qquad \left(\frac{\lambda_0}{\mu_0} + 2\right)C_m = -(m+1)D_m + bk'^2\Big[L_m - kL_{m-2} - \frac{n}{k'}\left(B_{m-2} - kB_{m-4}\right)\Big] + h\Big[\left(\frac{\lambda_0}{\mu_0} + 2\right)C_{m-2} + \frac{4(m-1)C_{m-2} + (m^3 + 2m^2 - 9m + 14)D_{m-2}}{(m-2)(m+3)}\Big].$$

It should be noted that, in the terms containing h, A_{m-2}, and B_{m-2} have been replaced by C_{m-2} and D_{m-2} by means of (108). In applying these equations, L_m and L_{m-2} are to be replaced by their values as given by (111), except in the case of L_2 which is to be eliminated by means of (114). In this way every C_m and D_m may be computed in terms of coefficients of lower order, and ultimately in terms of C_2, D_2, A_0.

For purposes of computation the formulas for C_m and D_m may conveniently be expressed explicitly in terms of like coefficients of the three preceding orders. This is possible when $m > 6$; when $m = 4$ and 6 special formulas are required. Convenient working formulas may be written as follows:

$$D_4 = \frac{bk'^2}{14}\left(C_2 - \frac{n}{k'}D_2\right) - \frac{k}{7}\Big[\left(\frac{\lambda_0}{\mu_0} + 2\right)C_2 + 3D_2 - 2hA_0\Big] + \frac{h}{7}(2C_2 + 5D_2),$$

$$(116)\qquad \left(\frac{\lambda_0}{\mu_0} + 2\right)C_4 = \frac{bk'^2}{14}\left(-4C_2 + 6D_2 - \frac{12k}{5}A_0 - \frac{n}{k'}C_2\right) - \frac{2k}{7}\Big[\left(\frac{\lambda_0}{\mu_0} + 2\right)C_2 + 3D_2 - 2hA_0\Big] + h\Big[\left(\frac{\lambda_0}{\mu_0} + 2\right)C_2 - \frac{4}{7}(C_2 - 3D_2)\Big].$$

$$D_6 = \frac{bk'^2}{36}\Big[C_4 - \frac{k}{35}(61C_2 + 30D_2) + \frac{12k^2}{35}A_0 - \frac{n}{k'}\left(D_4 - \frac{k}{7}(C_2 + 12D_2)\right)\Big] + \frac{h}{9}(C_4 + 8D_4),$$

$$(117)\qquad \left(\frac{\lambda_0}{\mu_0} + 2\right)C_6 = \frac{bk'^2}{36}\Big[-4C_4 + 6D_4 + \frac{k}{35}(286C_2 - 564D_2) + \frac{132k^2}{35}A_0 - \frac{n}{k'}\left(C_4 - \frac{k}{7}(11C_2 + 6D_2)\right)\Big] + h\Big[\left(\frac{\lambda_0}{\mu_0} + 2\right)C_4 - \frac{2}{9}(C_4 - 3D_4)\Big]\Big).$$

For $m = 8, 10, \cdots,$

$$D_m = bk'^2\left[\frac{C_{m-2}}{(m-2)(m+3)} - k\frac{2(4m^2-7m-41)C_{m-4}+60D_{m-4}}{5(m-4)(m-2)(m+1)(m+3)}\right.$$

$$+ k^2\frac{3(m^4-8m^3-m^2+76m+28)C_{m-6}+36(m^2-3m-4)D_{m-6}}{5(m-6)(m-4)(m-2)(m-1)(m+1)(m+3)}$$

$$\left.-\frac{n}{k'}\left(\frac{D_{m-2}}{(m-2)(m+3)} - k\frac{2C_{m-4}+(m^2-m-6)D_{m-4}}{(m-4)(m-2)(m+1)(m+3)}\right)\right]$$

$$+ h\left(D_{m-2} + \frac{4C_{m-2}-4D_{m-2}}{(m-2)(m+3)}\right),$$

$$\left(\frac{\lambda_0}{\mu_0}+2\right)C_m = bk'^2\left[\frac{-4C_{m-2}+6D_{m-2}}{(m-2)(m+3)}\right. \tag{118}$$

$$+ k\frac{4(8m^2-8m-97)C_{m-4}-12(4m^2-m-44)D_{m-4}}{5(m-4)(m-2)(m+1)(m+3)}$$

$$-k^2\frac{12(m^4-6m^3-14m^2+69m+46)C_{m-6} - 18(m^2-m-8)(m^2-3m-14)D_{m-6}}{5(m-6)(m-4)(m-2)(m-1)(m+1)(m+3)}$$

$$\left.-\frac{n}{k'}\left(\frac{C_{m-2}}{(m-2)(m+3)} - k\frac{(m^2-m-8)C_{m-4}+12D_{m-4}}{(m-4)(m-2)(m+1)(m+3)}\right)\right]$$

$$+ h\left[\left(\frac{\lambda_0}{\mu_0}+2\right)C_{m-2} - \frac{8C_{m-2}-24D_{m-2}}{(m-2)(m+3)}\right].$$

It is seen that, in (118), the governing terms as regards convergency are those of degree 0 in m. Thus the formulas ultimately approach the forms

$$D_m = hD_{m-2}, \qquad C_m = hC_{m-2}, \tag{119}$$

which indicate convergency if $h < 1$; this of course includes the case of uniform elasticity ($h = 0$).

It remains to satisfy (114) and the boundary conditions (103), L_2 being replaced by its value (110). These reduce to the following three equations, which, after substitution of the values of the summations, become linear equations in C_2, D_2, A_0:

$$\left(\frac{\lambda_0}{\mu_0}+2\right)C_2 + 3D_2 - 2hA_0$$

$$= bk'^2\left[\frac{9}{25}\left(\sum_2^\infty\frac{C_m-2D_m}{m} - \sum_2^\infty\frac{C_m-2D_m}{m+5}\right)\right.$$

$$\left.-\frac{3}{5}k\left(\sum_2^\infty\frac{C_m-2D_m}{m+2} - \sum_2^\infty\frac{C_m-2D_m}{m+5}\right) + \left(\frac{2}{5}-\frac{n}{2k'}\right)A_0 - \frac{c}{2k'}\right], \tag{120}$$

$$\left(\frac{\lambda_0}{\mu_0}+2\right)\sum_2^\infty C_m+\frac{4}{5}\sum_2^\infty\frac{C_m+3D_m}{m}-\frac{24}{5}\sum_2^\infty\frac{C_m-2D_m}{m+5}+2A_0=0,$$

$$\sum_2^\infty D_m+\frac{2}{5}\sum_2^\infty\frac{C_m+3D_m}{m}+\frac{8}{5}\sum_2^\infty\frac{C_m-2D_m}{m+5}+A_0=0.$$

The cases for which numerical results have been computed are all covered by the foregoing equations; the specific formulas for each case being easily written by assigning particular values to k, h, and n.

6. **Case of uniform density and elasticity.** For the purpose of comparison with previously known solutions, it is of interest to show the form assumed by the main working formulas in the case $h=0$, $k=0$, both for unrestricted compressibility and for $\lambda=\infty$.

(*A*) *Compressibility unrestricted.* Putting $h=0$, $k=0$, $k'=1$, equations (118) become

$$D_m=\frac{b}{(m-2)(m+3)}(C_{m-2}-nD_{m-2}),$$

(121)

$$\left(\frac{\lambda}{\mu}+2\right)C_m=\frac{b}{(m-2)(m+3)}(-(4+n)C_{m-2}+6D_{m-2}),$$

and hold for $m=4, 6, 8, \cdots$. Successive applications of these equations determine every C_m and D_m in terms of C_2 and D_2. These values are to be substituted in (120), of which the second and third are unchanged while the first simplifies as follows:

(122)

$$\left(\frac{\lambda}{\mu}+2\right)C_2+3D_2=b\left[\frac{9}{25}\left(\sum_2^\infty\frac{C_m-2D_m}{m}-\sum_2^\infty\frac{C_m-2D_m}{m+5}\right)+\left(\frac{2}{5}-\frac{n}{2}\right)A_0-\frac{c}{2}\right].$$

(*B*) *Incompressibility.* When $\lambda=\infty$ we proceed as indicated in the general solution, equations (63)–(69). Equations (121) then become

(123) $$D_m=-\frac{bnD_{m-2}}{(m-2)(m+3)},\qquad K_m=\frac{6bD_{m-2}}{(m-2)(m+3)},$$

while (120) reduce to the following:

$$K_2+3D_2=b\left[\frac{18}{25}\left(\sum_2^\infty\frac{D_m}{m+5}-\sum_2^\infty\frac{D_m}{m}\right)+\left(\frac{2}{5}-\frac{n}{2}\right)A_0-\frac{c}{2}\right],$$

(124)

$$\sum_2^\infty K_m+\frac{12}{5}\sum_2^\infty\frac{D_m}{m}+\frac{48}{5}\sum_2^\infty\frac{D_m}{m+5}+2A_0=0,$$

$$\sum_2^\infty D_m+\frac{6}{5}\sum_2^\infty\frac{D_m}{m}-\frac{16}{5}\sum_2^\infty\frac{D_m}{m+5}+A_0=0.$$

Static problem. When $n = 0$ equations (123) and (124) result in the well-known solution for the static strain of a homogeneous incompressible sphere. Thus equations (123) show that, if $n = 0$, every D_m vanishes except D_2, that every K_m vanishes except K_2 and K_4, and that

$$K_4 = \tfrac{3}{7} b D_2 . \tag{125}$$

Equations (124) are thus reduced to the following:

$$\begin{aligned} K_2 + \left(3 + \frac{9}{35} b\right) D_2 - \frac{2}{5} b A_0 + \frac{1}{2} bc &= 0, \\ K_2 + \left(\frac{18}{7} + \frac{3}{7} b\right) D_2 + 2A_0 &= 0, \\ \frac{8}{7} D_2 + A_0 &= 0 ; \end{aligned} \tag{126}$$

from which

$$\frac{A_0}{c} = \frac{8b}{38 + 4b}, \quad \frac{D_2}{c} = -\frac{7b}{38 + 4b}, \quad \frac{K_2}{c} = \frac{(2 + 3b)\,b}{38 + 4b}. \tag{127}$$

Also, from (107) and (108),

$$\begin{aligned} \frac{A_2}{c} &= \frac{3}{7} \frac{D_2}{c} = -\frac{3b}{38 + 4b}, \\ \frac{B_2}{c} &= \frac{5}{14} \frac{D_2}{c} = -\frac{5b}{2(38 + 4b)}, \\ \frac{B_0}{c} &= \frac{A_0}{2c} = \frac{4b}{38 + 4b}, \end{aligned} \tag{128}$$

so that the final solution is

$$\frac{e}{c} = \frac{8 - 3x^2}{38 \dfrac{\mu}{g\rho a} + 4}, \qquad \frac{\alpha}{c} = \frac{8 - 5x^2}{2\left(38 \dfrac{\mu}{g\rho a} + 4\right)}, \tag{129}$$

the known solution for the static problem.

Oscillation of an incompressible fluid sphere. It is of interest to note the simple form assumed by the solution in the case $\rho = \text{constant}$, $\lambda = \infty$, $\mu = 0$, $n \neq 0$. This is best treated by direct use of equations (44), (45), and (46), noting that $y = 0$ but $\lambda y \neq 0$, and that the second of (46) is satisfied by the condition $\mu = 0$. The simplified equations are the following:

$$\begin{aligned} x \frac{d(\lambda y)}{dx} &= ga \left(x \frac{dR}{dx} - n\rho e x^2 \right), \\ \lambda y &= ga\,(R - n\rho \alpha x^2), \end{aligned} \tag{130}$$

(131) $$R = \rho(e - \tfrac{3}{5}e_1 - \tfrac{1}{2}c)x^2,$$

(132) $$\lambda y = 0 \quad \text{when} \quad x = 1.$$

Equations (130) give

(133) $$x\frac{d(\alpha x^2)}{dx} - ex^2 = 0.$$

But since $y = 0$ the first of (48) gives

(134) $$6\alpha x^2 = \frac{d(ex^3)}{dx}.$$

Rejecting negative powers of x as irrelevant to the physical problem, equations (133) and (134) give

(135) $$e = 2\alpha = \text{constant} = e_1.$$

Hence

(136) $$R = \rho(\tfrac{2}{5}e_1 - \tfrac{1}{2}c)x^2,$$

(137) $$\lambda y = g\rho a\left(\left(\frac{2}{5} - \frac{n}{2}\right)e_1 - \frac{1}{2}c\right)x^2.$$

Finally, from the surface condition (132),

(138) $$\frac{e}{c} = \frac{e_1}{c} = \frac{5}{4 - 5n}.$$

Numerical applications of this equation will be given later.

7. **Quantities measuring the strain.** Two quantities are of especial importance because their values enter into the observational data from which the actual strain of the earth is inferred. One of these is the surface ellipticity e_1, while the other depends upon the change in the principal moments of inertia caused by the strain.

Surface ellipticity. This may be computed by the formula

(139) $$e_1 = \sum_{m=0}^{\infty} A_m = A_0 + \frac{2}{5}\sum_{m=2}^{\infty}\frac{C_m + 3D_m}{m} + \frac{3}{5}\sum_{m=2}^{\infty}\frac{C_m - 2D_m}{m + 5},$$

obtained by the use of (108).

Change of principal moments of inertia. Let C, A, A be the principal moments of inertia of the strained body (two of them being equal since S_2 has an axis of symmetry). Then it may be shown that

(140) $$A - C = -\frac{8\pi a^5}{15}\int_0^1 \rho\left(yx^4 - \frac{d(ex^5)}{dx}\right)dx,$$

which, by the use of (104), may be written

(141) $$A - C = \frac{16\pi a^5}{15}\int_0^1 \rho(e + 3\alpha)x^4\,dx.$$

Let I denote the diametral moment of inertia of the unstrained body, and let

$$f = \frac{A - C}{I}. \tag{142}$$

Then

$$I = \frac{8\pi}{3}\int_0^a \rho r^4\, dr = \frac{8\pi a^5}{3}\int_0^1 \rho x^4\, dx, \tag{143}$$

so that

$$f = \frac{2\int_0^1 \rho\,(e + 3\alpha)\,x^4\, dx}{5\int_0^1 \rho x^4\, dx}. \tag{144}$$

Using the value of ρ given by (94), and substituting the values of e and α from (105), the integrals in (144) become

$$\int_0^1 \rho\,(e + 3\alpha)\,x^4\, dx = \rho_0 \sum_{m=0}^{\infty} (A_m + 3B_m)\left(\frac{1}{m+5} - \frac{k}{m+7}\right), \tag{145}$$

$$\int_0^1 \rho x^4\, dx = \rho_0\left(\frac{1}{5} - \frac{k}{7}\right). \tag{146}$$

Convenience in the computations is gained by replacing A_m and B_m by C_m and D_m by means of (107) and (108), which give

$$A_0 + 3B_0 = \frac{5}{2}A_0, \qquad A_m + 3B_m = \frac{C_m + 3D_m}{m} \quad (m = 2, 4, \cdots), \tag{147}$$

so that (144) reduces to the form

$$\begin{aligned} f = A_0 + \frac{2}{5}\sum_2^{\infty}\frac{C_m + 3D_m}{m} - \frac{14}{5(7 - 5k)}\sum_2^{\infty}\frac{C_m + 3D_m}{m+5} \\ + \frac{2}{7 - 5k}\sum_2^{\infty}\frac{C_m + 3D_m}{m+7}. \end{aligned} \tag{148}$$

8. **Deflection of apparent gravity.** A quantity of importance in the application to the earth is the deflection of apparent gravity relative to the earth's surface due to the action of the disturbing forces.

The equation of the boundary surface of the strained body is

$$r = a\,(1 + e_1 S_2), \tag{149}$$

while that of a level surface may be written

$$r = a\,(1 + \mathrm{e}\, S_2). \tag{150}$$

The angle between the normals to the two surfaces (149) and (150) is

$$(\mathrm{e} - e_1)\frac{\partial S_2}{\partial \theta}. \tag{151}$$

The value of e may be found from the condition that the sum of the gravitation potential of the strained body and the disturbing potential is constant at the surface (150).

The change in the external gravitation potential due to the strain may be computed by the same method as that employed in finding the internal potential U; it is in fact expressed by the first integral in (35) if the upper limit r is replaced by a. Comparing with (140) it is seen that the change in the external potential is equivalent to

$$\frac{3\gamma(A - C)S_2}{2r^3} = \frac{3ga^2}{2r^3}\frac{IfS_2}{M}. \tag{152}$$

The external potential of the unstrained body is

$$\frac{\gamma M}{r} = \frac{ga^2}{r}, \tag{153}$$

and the potential of the disturbing forces is

$$W = \frac{cg}{2a}r^2 S_2. \tag{154}$$

The sum of these three parts, at the surface (150), to the first order of small quantities, is

$$ga\left[1 + \left(\frac{3If}{2Ma^2} + \frac{c}{2} - \mathrm{e}\right)S_2\right], \tag{155}$$

and this is constant if

$$\mathrm{e} = \frac{c}{2} + \frac{3If}{2Ma^2}. \tag{156}$$

This reduces (151) to

$$\left(\frac{c}{2} + \frac{3If}{2Ma^2} - e_1\right)\frac{\partial S_2}{\partial \theta}, \tag{157}$$

which is the angular deflection of apparent gravity with respect to the boundary surface.

Let q denote the ratio of this quantity to the value it would have if the body were absolutely rigid; then since the latter value is $\frac{1}{2}c\partial S_2/\partial\theta$,

$$q = 1 + \frac{3I}{Ma^2}\cdot\frac{f}{c} - 2\frac{e_1}{c}. \tag{158}$$

VI. Application to the Earth

1. **Values of constants in case of the earth.** In the case of the earth the mean radius and mean density are quite accurately known. Expressed in C. G. S. units the values here used are*

$$a = 6.371 \times 10^8, \qquad \rho_m = 5.527, \qquad \gamma = 6.6576 \times 10^{-8}.$$

From these result the values

$$g = 982, \qquad g\rho_m a = 3.458 \times 10^{12}.$$

As regards the variation of density within the earth, it is known that the surface density is about half the mean density, and that the diametral moment of inertia is about $\frac{5}{6}$ of the value for a homogeneous earth of density ρ_m; i.e.,

$$\rho_1 = \tfrac{1}{2}\rho_m, \qquad I = \tfrac{1}{3}Ma^2. \tag{159}$$

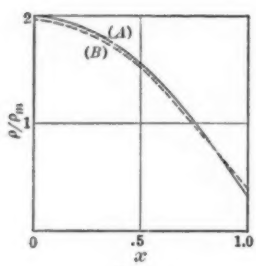

FIG. 1.

These equations can be satisfied only approximately by a density function of two terms, but the approximation can be sufficiently close to serve the present purpose. The formula which has been used is expressed by (94) with $k = \frac{5}{6}$. This gives a close approximation to the correct value of the moment of inertia, but makes the surface density less than that of actual surface rocks. A comparison with the well-known formula known as Laplace's law of density is given† in Table I and in Figure 1.

TABLE I. Comparison of Assumed Law of Density with Laplace's Law

$$(A) \quad \rho = \rho_0(1 - kx^2) = 2\rho_m(1 - \tfrac{5}{6}x^2),$$

$$(B) \quad \rho = \rho_0 \frac{\sin \kappa x}{\kappa x} = 1.955\rho_m \frac{\sin 2.461x}{2.461x}.$$

$x = \frac{r}{a}$	ρ/ρ_m	
	(A)	(B)
0	2.00	1.95
.2	1.93	1.87
.4	1.73	1.65
.6	1.40	1.31
.8	.93	.91
1.0	.33	.50

* The values of ρ_m and γ are those found by C. V. Boys, Nature, vol. 50, p. 419.

† The formula of Laplace satisfies both of equations (159) very closely. The same is true of the trinomial formula

$$\rho = \rho_0\left(1 - \frac{14}{13}x^2 + \frac{21}{65}x^4\right),$$

and numerical solutions using this formula are entirely practicable, but considerable additional labor would be imposed by the addition of a third term to the density formula.

The values of μ and λ have been determined experimentally for many specimens of surface rocks. Concerning the values throughout the interior of the earth the only evidence is the comparison of the actual yielding of the earth to disturbing forces with the computed yielding. Computations based on the assumption of uniform elasticity throughout the body have made it quite certain that the material of the earth is on the average far more rigid than the surface rocks. The computations which have now been made on the assumption of variable elasticity are based upon formulas (95) with values of the constants λ_0, μ_0, and h so taken that

$$\lambda_1 = \mu_1 = 2.542 \times 10^{11} \quad \text{C. G. S. units.}$$

These surface values are obtained by averaging a considerable number of results given by Adams and Coker.*

2. **Assumption regarding** n. The value of n for the actual tidal disturbance is so small that the terms containing n in the formulas expressing the solution may be neglected in computations relating to the problem of the rigidity of the earth. The problem of forced and free vibrations is, however, of some interest in itself, and one series of numerical results will be given based upon a series of values of n.

3. **Summary of numerical results for static problem.** The numerical results which have been obtained for the static problem fall into three main groups, the design being to show separately the influence of variable density and variable elasticity upon the quantities which are involved in estimates of the rigidity of the earth. The grouping corresponds to the following three assumptions:

(1) ρ and μ both constant;

(2) ρ variable, μ constant;

(3) ρ and μ both variable.

Groups (1) and (2) each consist of two series of results, one for $\lambda = \infty$ (incompressibility), the other for $\lambda = \mu$. The quantities of chief interest are e_1/c, f/c, and q, since it is these whose actual values are inferred from observation; and the results are so arranged as to show the way in which each of these quantities depends upon μ.

* *An investigation into the elastic constants of rocks, more especially with reference to cubic compressibility*, by Frank D. Adams and Ernest G. Coker, American Journal of Science, vol. 172 (1906), pp. 95–123. Averages of the values found for seventeen specimens of rocks of various kinds are as follows: $\mu = 2.542 \times 10^{11}$ dynes/cm², $\lambda + \frac{2}{3}\mu = 4.218 \times 10^{11}$ dynes/cm². These give $\lambda/\mu = 0.993$.

(1) *Case of uniform density and elasticity.* (*Table II and Figure 2*). Although the solution of this case is known,* the series of numerical results here given is more extended than has heretofore been published. Of especial

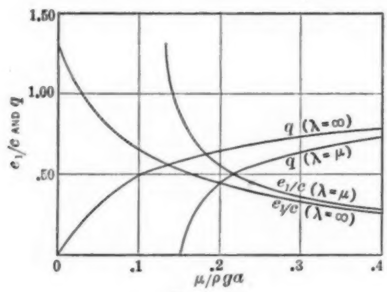

FIG. 2.

interest is the curve representing the results for the case $\lambda = \mu$. It is seen that this curve shows an infinite discontinuity for a value of $\mu/g\rho a$ between 0.12 and 0.125. This corresponds to the case of "gravitational instability" which is discussed by Love.† Algebraically this case results from the fact

TABLE II. Results for the Case of Uniform Density and Elasticity

$\frac{\mu}{\rho g a}$	$\lambda = \mu$			$\lambda = \infty$	
	$\frac{e_1}{c}$	$\frac{f}{c}$	q	$\frac{e_1}{c} = \frac{f}{c}$	q
0	1.250	1.250		1.250	0
.05	.732	.815		.847	.322
.10	.247	.554		.641	.487
.12	−2.371	−.030		.584	.533
.125	4.955	1.472		.571	.543
.15	.830	.577	.032	.515	.588
.20	.535	.449	.470	.431	.655
.30	.366	.331	.665	.325	.740
.40	.285	.264	.747	.260	.792
.50	.235	.220	.794	.217	.826

* *Some Problems of Geodynamics*, by A. E. H. Love. See especially Chapters VII and VIII. It is of interest to note the following results given by Love:

$$\frac{\lambda}{\mu} = \frac{46}{25} = 1.84, \qquad \frac{\mu}{g\rho a} = \frac{5}{24} = 0.208, \qquad \frac{e_1}{c} = 0.466, \qquad \frac{f}{c} = 0.4275;$$

$$\frac{\lambda}{\mu} = \frac{2041}{1944} = 1.05, \qquad \frac{\mu}{g\rho a} = 0.202, \qquad \frac{e_1}{c} = 0.522, \qquad \frac{f}{c} = 0.436.$$

It will be seen that these conform well to the results shown in Fig. 2. In making this comparison it should be noted that Love's symbols h, k are so defined that

$$h = 2\frac{e_1}{c}, \qquad k = \frac{3I}{Ma^2}\cdot\frac{f}{c},$$

so that the quantity above designated by q is, in Love's notation, $1 + k - h$.

† *Some Problems of Geodynamics*, Chapter IX.

that, for certain values of the elastic constants, equations (120) may be satisfied with $c = 0$ (i.e., with zero disturbing force) without the vanishing of the coefficients in the assumed developments of e and α.

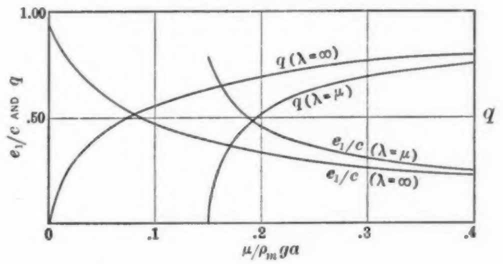

FIG. 3.

(2) *Case of variable density but uniform elasticity.* (*Table III and Figure 3*). The effect of the assumption of variable density is seen by comparing the two series of results in Table III and Figure 3 with the corresponding series

TABLE III. Results for the Case of Variable Density and Uniform Elasticity

(Values in brackets obtained by extrapolation.)

$\frac{\mu}{\rho_m ga}$	$\lambda = \mu$			$\lambda = \infty$		
	$\frac{e_1}{c}$	$\frac{f}{c}$	q	$\frac{e_1}{c}$	$\frac{f}{c}$	q
0	.942			.942	.910	0
.05				.597	.643	.430
.10				.472	.514	.555
.15	.763	.532	.010	.390	.426	.634
.20	.436	.395	.512	.333	.365	.689
.30	.296	.291	.690	.258	.283	.759
.40	.234	.236	.760	[.211]	[.231]	[.802]

in Table II and Figure 2. The variation of the strain with μ is similar in the two cases, but the strain for a given value of μ is materially less in the case of variable than in that of uniform density. The two curves for the case $\lambda = \mu$ resemble each other also in the feature of an infinite discontinuity.*

* The numerical solution in the case of variable density involves a considerable amount of labor; it is, however, greatly facilitated by the use of calculating machines. In preparation for the solution of a series of cases with specific values of the elastic moduli, the coefficients in formulas (116), (117), and (118) which depend upon k and m may be computed once for all for values of m as high as may be deemed necessary; these are used in connection with the numerical values of b and of λ/μ appropriate to each specific case. Every D_m and C_m is thus computed from terms of lower order, the resulting values being linear functions of C_2, D_2, and A_0. From these the summations occurring in equations (120) may be computed as linear functions of C_2, D_2, and A_0; it is then easy to complete the solution.

3. *Case in which both density and elasticity are variable.* (*Table IV and Figure 4*). A series of results for the case of variable elasticity was obtained by giving to the constants μ_0 and h in the formula $\lambda = \mu = \mu_0(1 - hx^2)$ four sets of values, so taken that μ_1 has always the value 2.542×10^{11} already cited as holding for surface rocks, while μ_0/μ_1 has the four values 4, 5, $6\frac{2}{3}$, 10. The four assumed values of h are 0.75, 0.80, 0.85, 0.90. Table IV and Figure 4 show the way in which the three quantities e_1/c, f/c, and q vary with μ_0.

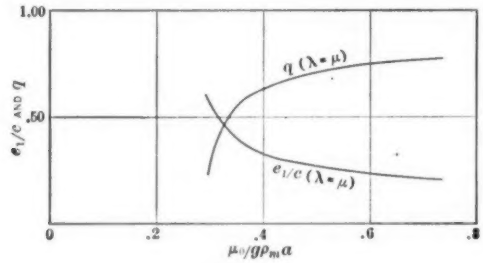

FIG. 4.

4. **Rigidity of the earth.** Experimental evidence as to the yielding of the earth to small disturbing forces involves the two quantities f/c and q.

The value of f/c may be inferred from the earth's free nutation period as determined by observed variations of latitude, the prolongation of the period

TABLE IV. Results for Variable Density and Variable Elasticity

h	$\frac{\mu_0}{\mu_1}$	b	$\frac{\mu_0}{\rho_m ga}$	$\frac{e_1}{c}$	$\frac{f}{c}$	q
.75	4	3.4025	.294	.605	.440	.217
.80	5	2.7220	.367	.374	.335	.576
.85	$6\frac{2}{3}$	2.0415	.490	.283	.269	.695
.90	10	1.3610	.735	.211	.207	.779

as compared with that for an absolutely rigid body being accounted for (as Newcomb first pointed out) by elastic yielding.* This line of evidence indicates a value of f/c between 0.27 and 0.29.

The observed effect of lunar and solar tidal forces upon the direction of gravity relative to the earth at a given place should theoretically give the

* It may be shown that the actual nutation period is to a close approximation the same as that of an unyielding body having the figure which the earth would assume if centrifugal forces were annulled. The theory leading to this result was given by the present writer in a paper presented to the American Mathematical Society, San Francisco Section, Dec. 20, 1902, of which an abstract was published in the Bulletin of the Society, vol. 9 (1903), p. 299. Similar reasoning was employed by J. Larmor in a paper in Proceedings of the Royal Society, London, Ser. A, vol. 82 (1909).

value of q. This quantity is, in fact, the ratio of the actual deflection of gravity to the deflection computed on the assumption that the earth is absolutely unyielding. Precise observations of this effect have been made by O. Hecker and others by the use of the horizontal pendulum, and by A. A. Michelson by measurement of the actual water tides in closed horizontal pipes. The observations of Hecker* gave values of q varying with azimuth, indicating a greater yielding of the earth in the north-south than in the east-west azimuth. Michelson's results,† however, indicate equal yielding in all azimuths, the value of q adopted as most probable being 0.69.

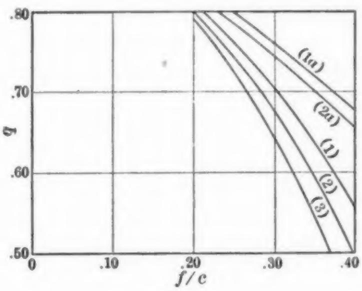

Fig. 5.

The values of f/c and q which are inferred from observation cannot be harmonized with theory on the assumption of homogeneity and incompressibility, and the discrepancy is only partly removed by assuming compressibility. The results above given show that both variable density and variable elasticity work in the direction of harmonizing the computed with the observed values. This is brought out by comparing the simultaneous values of f/c and q in each of the five series of results given in Tables II, III, and IV. The tabulated data are represented graphically in Figure 5, the five curves showing the relation between f/c and q for each of the five series. The curves (1*a*) and (2*a*) correspond to incompressibility, while (1), (2), and (3) are for the three cases in which $\lambda = \mu$.

Comparing the three curves (1), (2), and (3), it is seen that if f/c lies between

* O. Hecker, Veröffentlichungen des Kgl. Preussischen Institutes, No. 32, Berlin (1907).

† Michelson's preliminary observations are described in a paper entitled *Preliminary results of measurements of the rigidity of the earth*, Journal of Geology, vol. 22 (1914). See also Astrophysical Journal, vol. 39 (1914). These measurements gave 0.71 as the best value of q. (For a correction to the results originally published see Science, October 3, 1919, p. 327.) The value 0.69 was obtained from a more extended series of measurements, the account of which has not at this date (October, 1919) been published. The writer is indebted to Professor Michelson and Professor Henry G. Gale for the communication of the general result in advance of publication.

0.27 and 0.29 (as indicated by latitude observations), q should fall within the limits

0.74 and 0.72 in case (1), 0.72 and 0.69 in case (2), 0.69 and 0.66 in case (3).

Materially greater values of q are indicated by the curves (1a) and (2a).

Considering all the computations that have been made, the best agreement with the facts of observation results from the assumptions

$$\rho = \rho_0(1 - \tfrac{5}{6}x^2), \qquad \lambda = \mu = \mu_0(1 - .85x^2),$$

the value of ρ_0 being based upon the known value of ρ_m, and that of μ_0 being such as to make μ_1 agree with the value found experimentally for surface rocks. This surface value is

$$\mu_1 = 2.542 \times 10^{11} \text{ dynes/cm}^2,$$

giving for the value at the center

$$\mu_0 = \mu_1/.15 = 1.695 \times 10^{12} \text{ dynes/cm}^2.$$

5. **Forced oscillation.** The numerical results given above are for the static problem, all terms in the formulas which contain n being regarded as negligible. Inspection of equations (118) shows, however, that the same routine of computation can be followed when n has any known value. It is thus possible to determine the forced oscillation due to a disturbing potential of any known period.

6. **Free oscillation.** If an oscillation of the type assumed in the solution is possible in the absence of disturbing forces, equations (120) must be satisfied with $c = 0$. Obviously this is possible only for particular values of n, and the only method which suggests itself of determining such particular values is by interpolation in a series of results computed with assumed values of n. This method, though somewhat laborious, is entirely feasible, and has actually been carried out in one case with results which will be given.

If we consider a continuous series of forced oscillations with the same intensity of disturbing force but with period decreasing from ∞ (corresponding to static strain), the period of possible free oscillation will be that at which the forced oscillation reverses its phase while its amplitude passes through ∞. If, therefore, for each of a series of values of n, beginning with 0, the values of C_2/c, D_2/c, A_0/c are computed, each of these will increase in magnitude with increasing n, approaching positive or negative infinity as n approaches the value corresponding to the free oscillation, then changing sign and decreasing in magnitude. The required critical value of n may be found by interpolation in one of these series, preferably by taking reciprocals and interpolating for the zero-value.

The case for which computations have been made is that in which the density and elastic moduli are represented by equations (94) and (95) with $k = \frac{5}{6}$ and $h = 0.85$. Results for $n = 0$ have already been given in Table IV, and further computations have been made for $n = 1, 2$, and 3, giving the results shown in Table V.

TABLE V. Results for Case of Forced Oscillation

[Results in brackets obtained by interpolation for case of free vibration.]

n	Period in seconds.	$\frac{C_2}{c}$	$\frac{D_2}{c}$	$\frac{A_0}{c}$	$\frac{e_1}{c}$	$\frac{c}{e_1}$
0	∞	.795	$-$ .540	.381	.283	3.53
1	5060	1.668	$-$ 1.146	.608	.436	2.29
2	3578	6.820	$-$ 4.687	1.492	.948	1.06
[2.77]	[3040]	[$\pm\infty$]	[$\pm\infty$]	[$\pm\infty$]	[$\pm\infty$]	[0]
3	2922	$-$ 44.04	$+$ 30.90	$-$ 5.857	$-$ 2.93	$-$.34

The interpolation for the critical value of n may be accomplished to a good approximation by means of the graph (A) shown in Figure 6, which represents c/e_1 as a function of n; the zero-value falling at about $n = 2.77$.

From the meaning of p as given by (27), it is obvious that the period of a complete oscillation is $2\pi/p$ or $2\pi\sqrt{a/ng}$. Putting $a = 6.371 \times 10^8$ cm., $g = 982$ cm/sec^2, the period in seconds is $5060/\sqrt{n}$. The period corresponding to $n = 2.77$ is thus about 3040 seconds.

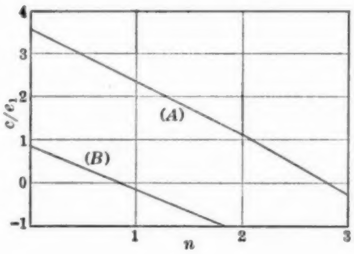

FIG. 6.

It is of interest to compare this result with that given by equation (138) for a homogeneous incompressible fluid sphere. The critical value of n for this case is 0.8, corresponding to a free oscillation period of 5650 seconds. The graph of c/e_1 as a function of n is the straight line (B) shown in Figure 6.

THE GEOMETRY OF HERMITIAN FORMS*

BY

JULIAN LOWELL COOLIDGE

1. The general case

We mean by a *hermitian form* an expression which is linear and homogeneous in each of two sets of conjugate imaginary variables and which is a constant multiple of its own conjugate imaginary. Such a form may be written

$$(1) \qquad \sum a_{ij} x_i \bar{x}_j \qquad (i, j, = 0, 1, \cdots n),$$

where

$$a_{ij} = \rho \bar{a}_{ji}, \qquad \rho^2 = 1, \qquad |a_{ij}| \neq 0.$$

The last inequality is added for convenience, and is not needed in all parts of the work. Strictly speaking, we are dealing with hermitian forms of non-vanishing discriminant.

It is easy to write down a hermitian form in the Clebsch-Aronhold symbolism. We here replace (1) by

$$(2) \qquad a_x \bar{a}_{\bar{x}}.$$

For the discriminant we have the expression

$$\frac{1}{(n+1)!} \left| a^{(1)} a^{(2)} \cdots a^{(n+1)} \right| \cdot \left| \bar{a}^{(1)} \bar{a}^{(2)} \cdots \bar{a}^{(n+1)} \right|,$$

each of the last two factors being a symbolic $(n+1)$-rowed determinant.

In order to live up to the geometric title of our paper, we shall speak of a set of $(n+1)$ homogeneous values (x), not all simultaneously zero, as the homogeneous coördinates of a *point* in a projective space of n-dimensions. Following the conventions of this sort of geometry, we shall define the system of all points whose coördinates satisfy a linear homogeneous equation

$$(3) \qquad \sum_i u_i x_i \equiv (ux) = 0 \qquad (i = 0, 1, \cdots n)$$

as forming a *hyperplane*, or S_{n-1}. If there be k such equations, linearly independent, the corresponding points shall be said to generate an S_{n-k}. The coördinates of all points of the system are linearly dependent on those of any

* Presented to the Society, December 30, 1919.

$n-k+1$ linearly independent individuals of their number. Conversely, such dependence will always lead back to k independent equations, and an S_{n-k}.

Two points (x) and (y) whose coördinates are connected by one and, hence, all of the equations

$$\sum_{i,j} a_{ij}\, x_i\, \bar{y}_j = \sum_{i,j} a_{ji}\, y_j\, \bar{x}_i = a_x\, \bar{a}_{\bar{y}} = a_y\, \bar{a}_x = 0 \tag{4}$$

are said to be *conjugate* with regard to the form (1). The totality of points conjugate to a given point will generate a hyperplane which shall be defined as *polar hyperplane* of the given point. More generally, we may state

THEOREM I. *All points of an S_{n-k} are conjugate to all of an S_{k-1} with regard to a hermitian form.*

Let us take any point whose coördinates do not reduce the form (1) to zero. Such a point will not lie in its polar hyperplane, will not be self-conjugate, and may be chosen in ∞^n ways. A second point may then be chosen conjugate to the first but not to itself, then a third conjugate to the first two, etc. We arrive finally at a system of points, each two of which are conjugate with regard to the form. Remembering the identity

$$n+(n-1)+\cdots-+2+1=\tfrac{1}{2}\,n(n+1),$$

we have

THEOREM II. *There are $\infty^{\frac{1}{2}(n(n+1))}$ systems of $(n+1)$ points, each two of which are conjugate with regard to a given hermitian form of a non-vanishing discriminant.*

Let us take such a system of points as the basis of our coördinate apparatus; we find immediately that our form takes the canonical shape

$$\sum_i a_{ii}\, x_i\, \bar{x}_i, \tag{5}$$

where all of the coefficients are real. Evidently, when the discriminant is zero, there is a similar canonical expression, some of the coefficients vanishing. A further reduction is effected by the transformation

$$x_i = \sqrt{|a_{ii}|}\,x_i.$$

THEOREM III. *Every hermitian form of non-vanishing discriminant may be reduced in $\infty^{\frac{1}{2}(n(n+1))}$ ways to an expression of the type*

$$\sum_i \pm\, x_i\, \bar{x}_i. \tag{6}$$

It remains to be seen what is the significance of the distribution of positive and negative signs. Suppose that things are so arranged that the first k

terms in (6) have one sign, while the remaining terms have the other sign, and that $k \leqq n - k + 1$.

Consider the S_{k-1} given by the equations

$$x_0 - x_k = x_1 - x_{k+1} = \cdots x_{k-1} - x_{2k-1} = 0,$$

$$x_{2k} = x_{2k+1} \cdots = x_n = 0.$$

Every point of this S_{k-1} lies in its polar hyperplane with regard to the form. On the other hand there could not be an S_k each of whose points was in its polar hyperplane. For if there were such a variety, that would have at least one point in common with the S_{n-k} given by the equations

$$x_0 = x_1 = \cdots x_{k-1} = 0,$$

and for such a point (x) we should have the absurd equation

$$x_k \bar{x}_k + x_{k+1} \bar{x}_{k+1} + \cdots + x_n \bar{x}_n = 0.$$

Our reasoning here is reversible throughout. We thus reach

THEOREM IV. *There will exist an S_{k-1} all of whose points are self-conjugate with regard to a given hermitian form of non-vanishing discriminant, but no S_k possessing this property when $k \leqq \frac{1}{2} n$, and when the form can be reduced to a sum of products of conjugate imaginary coördinate values, whereof just k products have one algebraic sign, and the remainder the other sign.*

THEOREM V. *Sylvester's law of inertia holds for hermitian forms.**

2. Pseudo-orthogonal relations

We shall, from now on, confine ourselves to the special case of the hermitian form

$$\sum_i x_i \bar{x}_i \equiv (x\bar{x}). \tag{7}$$

Since this expression can never vanish, there are no points which are self-conjugate. An S_k and an S_{n-k+1}, so related that each point of one is conjugate to each of the other shall be said to be *pseudo-orthogonal*. The same unwieldy adjective shall be applied to every transformation that leaves the form unaltered except for a constant non-vanishing factor. If such a transformation be a collineation of the type

$$x_i' = \sum_j a_{ij} x_j \qquad (|a_{ij}| \neq 0), \tag{8}$$

the fundamental equations of condition are

* Given without proof by Segre in the paper, *Un nuovo campo di ricerche geometriche*, Atti della R. Accademia delle Scienze di Torino, vol. 25 (1890), p. 605.

$$\sum_k a_{kp}\, \bar{a}_{kq} = 0 \qquad (p \neq q), \tag{9a}$$

$$\sum_k a_{kp}\, \bar{a}_{kp} = \sum_k a_{kq}\, \bar{a}_{kq}. \tag{9b}$$

Let us see what equations (9*a*) mean geometrically, as that will lead us to discover an explicit form for our pseudo-orthogonal collineation. They tell us that the members of two different columns of the matrix of the collineation are proportional to the coördinates of two points which are conjugate with regard to the form. Equations (9*b*) are satisfied by affecting such coördinates with proper factors of proportionality.

The first point whose coördinates shall go into this matrix may be selected at random. The second may be chosen linearly dependent on the first and on an arbitrary point, yet conjugate to the first. The third may be taken linearly dependent on the first two and another independent arbitrary point but conjugate to the first two, and so on. The ith row of the matrix of the transformation would thus take the general form

$$\| a_i \quad pa_i + qb_i \quad ra_i + sb_i + tc_i \quad \cdots \|.$$

It is our business to choose the multipliers in such a way that the point whose coördinates go to make up any column is conjugate to those which appear in the columns to the left. Since the condition that two points (x) and (y) should be conjugate takes the form

$$(x\bar{y}) = (y\bar{x}) = 0, \tag{10}$$

the coördinates of the point which is linearly dependent on $(a)(b)\cdots(k)(l)$ may be expressed in the form

$$\begin{vmatrix} a_i & b_i & \cdots & k_i & l_i \\ (a\bar{a}) & (b\bar{a}) & \cdots & (k\bar{a}) & (l\bar{a}) \\ (a\bar{b}) & (b\bar{b}) & \cdots & (k\bar{b}) & (l\bar{b}) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (a\bar{k}) & (b\bar{k}) & \cdots & (k\bar{k}) & (l\bar{k}) \end{vmatrix}.$$

Our equations (9*a*) are thus completely satisfied. If we indicate by (y) the coördinates of a point appearing in any column, we have merely to multiply them all through by a factor η where

$$|\eta| = \frac{r}{\sqrt{(y\bar{y})}}.$$

If thus

$$\Delta_l = \begin{vmatrix} a & b & \cdots & k & l \\ (a\bar{a}) & (b\bar{a}) & \cdots & (k\bar{a}) & (l\bar{a}) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (a\bar{k}) & (b\bar{k}) & \cdots & (k\bar{k}) & (l\bar{k}) \end{vmatrix} \tag{11}$$

we reach our final statement:

THEOREM VI. *The general term of the matrix of a pseudo-orthogonal collineation can be expressed in the form*

$$(12)\qquad re^{i\theta_l}\frac{\begin{vmatrix} a_i & b_i & \cdots & k_i & l_i \\ (a\bar{a}) & (b\bar{a}) & \cdots & (k\bar{a}) & (l\bar{a}) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (a\bar{k}) & (b\bar{k}) & \cdots & (k\bar{k}) & (l\bar{k}) \end{vmatrix}}{\sqrt{\sum_{p,q}(p\bar{q})\dfrac{\partial \Delta_l}{\partial p}\dfrac{\partial X_l}{\partial \bar{q}}}}.$$

Here the subscript i indicates the row in which this term lies, while the order of the determinant Δ_l as defined in (11) *gives the column. The summation in the denominator covers all pairs of letters, one from each of the series $a, b, \cdot k, l$; $\bar{a}, \bar{b}, \cdots \bar{k}, \bar{l}$. The non-vanishing factor r is the same in every term.**

It is pleasant to note that the most important properties of a pseudo-orthogonal collineation are found with much less labor than is needed to establish the fundamental theorem. Let us look at the fixed points. Every collineation of space surely leaves at least one point fixed. That point can not be self-conjugate with regard to the fundamental hermitian form, and its polar hyperplane, which does not include the given point, is also fixed, the transformation therein being of exactly the same type in one less dimension.

THEOREM VII. *In every pseudo-orthogonal collineation there are necessarily $n+1$ fixed points, each two of which are conjugate with regard to the fundamental form.*

It is clear that we can take these fixed points as the basis of our coördinate system. We have then the canonical form for a pseudo-orthogonal collineation

$$(13)\qquad x'_k = re^{i\theta_k}x_k.$$

If $\theta_0 = \theta_1 = \cdots = \theta_p$, all points of the S_p, $x_{p+1} = x_{p+2} = \cdots = x_n = 0$ are invariant.

If $(a^{(0)})(a^{(1)})(a^{(2)})\cdots(a^{(n)})$ be $n+1$ distinct points, each two of which are conjugate with regard to our form, the ith row of a pseudo-orthogonal

* The only attempt that has been made to find the form for a pseudo-orthogonal collineation, so far as the writer is aware, is to be found in an article by Loewy, "*Ueber bilineare Formen mit conjugirt imaginaeren Variabeln,*" Abhandlungen der Kaiserlichen Leopoldinischen-Carolinischen Akademie, Halle, vol. 71 (1898). Loewy does not, however, give the transformation in explicit shape, but in a form that involves the solution of a certain matrix equation. It is to be noted that the number of apparently arbitrary quantities is really greater than that of the free parameters in the group of these transformations. It would be highly desirable to find an expression for the transformation in a form that involves no redundant parameters, the present writer regrets that all attempts which he has made so far have met with meager success.

collineation which transfers them to the fundamental points of the coördinate system will be

$$\left\| \frac{a^{(0)}}{\sqrt{(a^{(0)}\bar{a}^{(0)})}} \quad \frac{a_i^{(1)}}{\sqrt{(a^{(1)}\bar{a}^{(1)})}} \cdots \frac{a_i^{(n)}}{\sqrt{(a^{(n)}\bar{a}^{(n)})}} \right\|.$$

Taking the product of this, and the inverse of a second such transformation which carries a similar set of points (b) into the fundamental set, we have a pseudo-orthogonal transformation carrying the set (a) into the set (b).

THEOREM VIII. *A pseudo-orthogonal collineation may be found to carry any set of points, mutually conjugate with regard to the fundamental form, into any other such set.*

We are now in a position to determine the number of parameters of the pseudo-orthogonal group. To begin with, the determination of the $n+1$ points which shall be carried into the fundamental points depends upon

$$n + (n-1) + \cdots 2 + 1 + 0 = \tfrac{1}{2}n(n+1)$$

complex parameters. On the other hand, we see by (13) that the general collineation which leaves the fundamental points invariant depends on n real parameters, $\theta_k - \theta_0$, the common factor r being irrelevant.

THEOREM IX. *The totality of pseudo-orthogonal collineations is a group depending on $\frac{1}{2}n(n+1)$ complex, and n real parameters.**

Let us see if there be any one-parameter groups of pseudo-orthogonal transformations. Since all transformations of such a group have the same fixed points, we may take these as the basis of the coördinate system; it becomes a question of finding one-parameter groups under (13). Let us first look for groups depending on a single complex parameter. We may, without loss of generality, take $re^{i\theta_0}$ for the independent variable, and write

$$re^{i\theta_k} = f_k(re^{i\theta_0}).$$

But in this case the function

$$\frac{f_k(z)}{z}$$

has a constant modulus and so is a constant. Hence the common factor $re^{i\theta_0}$ could be divided out of (13) and the group would not depend on any parameter.

THEOREM X. *There are no infinite groups of pseudo-orthogonal collineations depending analytically on a single complex parameter.*

The case is different when we come to consider groups depending on a single real parameter. We may take θ_0 for this parameter, and write

$$\theta_k = f_k(\theta_0).$$

* Study, *Kürzeste Wege im komplexen Gebiete*, Mathematische Annalen, vol. 60 (1905), p. 323.

The fundamental property of groups gives us

$$f_k(r) + f_k(s) = f_k(r+s), \qquad f_k(t) = a_k t.$$

Dividing out the constant factor r, we get the general form for a transformation of this group

$$x'_k = e^{ia_k t} x_k.$$

An infinitesimal transformation of the group will give

$$dx_k = ia_k x_k \, dt.$$

Integrating, we get the form for the "threads," that is the path-curves of points under the group, namely

$$x_k = \rho_k e^{ia_k t}. \tag{14}$$

When will two pseudo-orthogonal collineations be commutative? Let us pick a transformation of (8) and one of (13). The conditions for commutativity are

$$(e^{i\theta_i} - e^{i\theta_j}) a_{ij} \equiv 0 \qquad (i, j, = 0, 1 \cdots n).$$

If one of our transformations, which we assume to be (13), has only $n+1$ fixed points, this equation can only be satisfied by taking

$$a_{ij} = 0, \qquad i \neq j,$$

and the points fixed for the first transformation are also fixed for the second. Suppose, however, that our transformation (13) has an S_p fixed, so that

$$\theta_0 = \theta_1 \cdots = \theta_p.$$

Then

$$a_{i(p+r)} = a_{(p+s)j} = 0.$$

The first $p+1$ equations of (8) can, then, be written

$$\begin{aligned} x''_0 &= a_{00} x'_0 + a_{01} x'_1 + \cdots a_{0p} x'_p, \\ x''_1 &= a_{10} x'_1 + \cdots, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x''_p &= a_{p0} x'_0 + \cdots a_{pp} x'_p. \end{aligned}$$

These, in turn, can be reduced to

$$x''_0 = e^{i\phi_0} x'_0, \qquad x''_1 = e^{i\phi_1} x_1, \qquad \cdots, \qquad x''_p = e^{i\phi_p} x'_p,$$

while the first $p+1$ transformations of (8) remain

$$x'_0 = x_0, \qquad x'_1 = x_1, \qquad \cdots, \qquad x'_p = x_p.$$

We are thus enabled to state the general theorem:

THEOREM XI. *A necessary and sufficient condition that two pseudo-orthogonal collineations should be commutative is that every S_p all of whose points are invariant in the first transformation, but which is not contained in an S_{p+1} of fixed points, should be invariant in the other transformation.*

It is easy enough to find relative invariants under pseudo-orthogonal collineations. For instance, the points $(a)(b)\cdots(k)$ will have the relative invariant

$$\begin{vmatrix} (a\bar{a}) & (a\bar{b}) & \cdots & (a\bar{k}) \\ (b\bar{a}) & \cdots & \cdots & (b\bar{k}) \\ \cdot & \cdot & \cdot & \cdot \\ (k\bar{a}) & \cdots & \cdots & (k\bar{k}) \end{vmatrix}.$$

This can vanish only when the points are linearly dependent, since it is the product of the two conjugate imaginary matrices

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \\ \cdot & \cdot & \cdot & \cdot \\ k_0 & k_1 & \cdots & k_n \end{vmatrix} \times \begin{vmatrix} \bar{a}_0 & \bar{a}_1 & \cdots & \bar{a}_n \\ \bar{b}_0 & \bar{b}_1 & \cdots & \bar{b}_n \\ \cdot & \cdot & \cdot & \cdot \\ \bar{k}_0 & \bar{k}_1 & \cdots & \bar{k}_n \end{vmatrix}.$$

Let us take two points; remembering that the product of two conjugate imaginary expressions is necessarily positive, we see that

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{vmatrix} \times \begin{vmatrix} \bar{a}_0 & \bar{a}_1 & \cdots & \bar{a}_n \\ \bar{b}_0 & \bar{b}_1 & \cdots & \bar{b}_n \end{vmatrix} > 0,$$

$$(a\bar{a})(b\bar{b}) > (a\bar{b})(b\bar{a}).$$

We are thus lead to a new system of non-euclidean geometry where the distance of two points (a) and (b) is defined by the equation

$$\cos\frac{d}{k} = \frac{\sqrt{(a\bar{b})}\,\sqrt{(b\bar{a})}}{\sqrt{(a\bar{a})}\,\sqrt{(b\bar{b})}}. \tag{15}$$

These considerations of hermitian metrics are, however, outside the domain of our present paper; they have, in fact, already been treated by others.*

* Study, loc. cit., p. 333; also Fubini, *Sulle metriche definite da una forma hermitiana*, Atti della R. Istituto Veneto, vol. 63 (1903).

AMERICAN EXPEDITIONARY FORCES,
May, 1918.

CERTAIN TYPES OF INVOLUTORIAL SPACE TRANSFORMATIONS*

BY

F. R. SHARPE AND VIRGIL SNYDER

1. Introduction

1. Involutorial point transformations of space that have a surface of invariant points can be derived from $(2, 1)$ correspondences between two spaces (x) and (x'), namely, the transformation which interchanges the two points in (x) that correspond to the same point in (x'). The correspondence between (x) and (x') may be expressed algebraically by three equations reducible to the form

$$\rho x_i = \phi_i(x_1, x_2, x_3, x_4) = \phi_i(x) \qquad (i = 1, 2, 3, 4).$$

In our preceding paper on this subject† it was supposed that one of the equations expressing the correspondence was bilinear in (x) and (x'). This restriction is here removed. The purpose of the present paper is to establish a general method of determining the basis elements of the web of surfaces

$$\sum_{i=1}^{4} a_i \phi_i = 0$$

and to enumerate the possible types of associated involutions, when the order of the surfaces of the web is not greater than five.

It is shown that each surface of a web of quartics through a curve of order 8 and genus 2 is invariant under two distinct involutions whose product is discontinuous and of infinite order. These surfaces are similar to, but distinct from, the Fano quartics through a curve of order six and genus 2. Other interesting cases are those in which the surfaces of branch-points L' in (x') are quartics. The focal surface of every line congruence of order two appears among them, and a number of others which together form an uninterrupted chain, having common properties. The surfaces K of coincident points in (x) are birationally equivalent to L', and have a different system of interesting properties. When the surfaces of the web are quadrics, L' is the

* Presented to the Society, September 3, 1919.

† These Transactions, vol. 20 (1919), pp. 185–202.

sixteen nodal Kummer surface and K the Weddle surface. This case is well known* but the other cases are believed to be new.

The problem reduces to that of finding a web of surfaces $\sum a_i \phi_i = 0$ having the property that any three surfaces of the web intersect in two variable points. The method has two advantages: first, the involutorial character of the correspondence between the two points (x) is assured; second, certain properties can be more easily discovered in (x'), and then interpreted in (x).

2. **General formulas.** For convenience of reference the following formulas, due chiefly to Noether† are here collected. If two surfaces F_{n_1}, F_{n_2} of orders n_1, n_2 contain a curve C_m of order m and rank r to multiplicities i_1, i_2, then the residual intersection C_μ meets C_m in

$$(1) \qquad \tau = m(i_2 n_1 + i_1 n_2 - 2 i_1 i_2) - i_1 i_2 r$$

points and has the genus

$$(2) \qquad \mu(n_1 + n_2 - 4) - (i_1 + i_2 - 1)\tau + 1 .$$

If F_{n_1}, F_{n_2} have C_m to multiplicities i_1, i_2 and also $C_{m'}$ to multiplicities i_1', i_2', and C_m meets $C_{m'}$ in s points, then C_μ meets C_m in $\tau - i_1' i_2' s$ points, and $C_{m'}$ in $\tau' - (i_1 i_2' + i_2 i_1' - i_1' i_2') s$ points, provided $i_1 \geqq i_1'$, $i_2 \geqq i_2'$.

If C_m is i_3-fold on a third surface F_{n_3}, then C_m is equivalent to E_m points of intersection of the three surfaces, where

$$(3) \qquad E_m = m(i_2 i_3 n_1 + i_3 i_1 n_2 + i_1 i_2 n_3 - 2 i_1 i_2 i_3) - r i_1 i_2 i_3 .$$

The equivalence of C_m and $C_{m'}$ is

$$(4) \qquad E_m + E_{m'} - (i_2' i_3' i_1 + i_3' i_1' i_2 + i_1' i_2' i_3 - i_1' i_2' i_3') s .$$

The postulation of C_m, i-fold on F_n, is

$$(5) \qquad P_m = \frac{i(i+1)}{2}\left(m(n+2) - \frac{2i+1}{6}(r + 2m) \right).$$

For C_m and $C_{m'}$, i-fold, i'-fold respectively, the postulation is

$$(6) \qquad P_m + P_{m'} - \frac{i'}{2}(i' + 1)(3i - i' + 1) s .$$

* Hudson, *Kummer's Quartic Surface*, Chapter XV.

V. Eberhard, *Ueber eine räumlich involutorische Verwandtschaft, 7. Grades . . .*, Breslau Dissertation, 1885.

Snyder, *An application of the* (1, 2) *quaternary correspondence to the Kummer and Weddle surfaces*, these Transactions, vol. 12 (1911), pp. 354–366.

† Noether, *Sulle curve multiple di superficie algebriche*, Annali di Matematica, series 2, vol. 5 (1871), pp. 163–177.

The postulation of an l-fold point through which pass j branches of an i-fold curve is

(7) $$\frac{l}{6}(l+1)(l+2)-\frac{i}{6}(i+1)(3l-2i+2)j.$$

The genus of the variable curve of intersection of two surfaces of the web can also be found from the Riemann-Roch theorem for surfaces. If we assume the surfaces $\phi_i(x)=0$ of the web to be regular, we have

(8) $$r=p_a+n-\pi+1,$$

in which $r=2$, the dimensionality of the system of curves on a fixed surface of the system, p_a is the arithmetic genus of the surface, $n=2$, the number of points in which two curves of the system intersect (grade of the system), and π is the genus of the curve. Hence the

THEOREM. *The genus of a variable curve of intersection of two surfaces of the web is one greater than the arithmetic genus of a general surface of the web.*

Moreover the order of L', the surface of branch-points in (x'), is $2\pi+2$. A plane in (x) meets K in a plane curve (s_1, K). The image of s_1 is a surface s_n'; the image of K is L'. The image of (s_1, K) is the contact curve of s_n' and L'.

The image s_N of s_1 in the involution I in (x) passes through (s_1, K). The residual intersection of s_1 with s_N has for image a double curve on s_n'. The basis points and fundamental elements will be discussed in connection with each case.

3. **Simple basis curve.** If the surfaces of the web are of order n and have a simple basis curve C_m of genus p a d have ξ simple basis points, we have from (3)

$$(3n-4)m-2p+2+\xi=n^3-2,$$

and from (5)

$$nm-p+1+\xi=\frac{(n+1)(n+2)(n+3)}{6}-4.$$

4. **Quadrics.** If $n=2$ we have the following cases.

A. $m=0$, $\xi=6$. B. $m=1$, $\xi=2$. C. $m=2$, $p=0$, $\xi=0$.

The case A is the well-known correspondence in which K is the Weddle surface and L' the Kummer surface. In case C it can be shown that a line joining conjugate points in (x) passes through a fixed point so that the involution is of the monoidal type already considered by Montesano.*

Case B can be transformed into the special case of C in which the basis conic consists of two intersecting lines. A similar case exists in which the quadrics have 2 basis points and also touch a fixed plane at a fixed point.

* Montesano, *Su le trasformazioni involutorie monoidali*, Istituto Lombardo, Rendiconti, series 2, vol. 21 (1888), pp. 579–594.

2. Webs of cubics

5. From the preceding formulas we have

$$5m - 2p + 2 + \xi = 25, \qquad 3m - p + 1 + \xi = 16,$$

and find the following possible forms of basis elements.

A. Three mutually skew lines α_i and four points P_i.

B. Rational quartic curve β_4 and three points P_i.

C. Quintic curve β_5 of genus 2 and two basis points P_i.

D. Sextic curve β_6 of genus 4, and one point P.

These four cases will be considered in turn.

6. **Case A.** The image of a plane s_1' in (x') is a cubic surface s_3 in (x), passing through the three lines α_i and each of the points P_i. A line c_1' of (x') is transformed into a sextic curve c_6, the residual intersection of two cubic surfaces of the web. It meets each line α_i in 4 points and passes simply through each point P_i. The image of a point on α_i is a straight line in (x'). As the point describes α_i the image line describes a rational ruled surface A_4' of order 4. The images of the points P_i are planes π_i'. A plane s_1 has a sextic surface s_6' for image, and a line c_1 has a space cubic curve c_3' for image. A line meeting α_i has a fundamental line, generator of A_4', and a conic for images. The lines meeting $\alpha_1, \alpha_2, \alpha_3$ have points of a curve ρ' for images. The curve ρ' is therefore the image of the quadric R_2 determined by $\alpha_1, \alpha_2, \alpha_3$. Since every plane s_1 in (x) meets every generator of R_2 in one point, the image surface s_6' contains ρ' as a simple curve. Moreover, R_2 meets every cubic surface s_3 of the web in 3 generators, hence ρ' is a space cubic curve. A straight line c_1 meets R_2 in 2 points, hence the image cubic c_3' meets ρ_3' in 2 points. The image of any line of R_2 belonging to the same regulus as α_i is ρ_3' itself.

A line through P_i has for image a conic and a line in π_i'. Through P_i passes one line $g_{i,kl}$ meeting α_k, α_l. The image of this line consists of three fundamental lines and a point $G_{i,kl}'$. This point lies on all the sextic surfaces s_6' since s_1 meets $g_{i,kl}$ in a point. In (x) are 12 such lines g and in (x') are 12 such points G'.

An s_1' through ρ_3' has for images in (x) the quadric R_2 and a surface s_4 through $\alpha_1, \alpha_2, \alpha_3$, having double points at each P_i. Two quadrics through ρ_3' meet in a bisecant of ρ_3'. The image quartics s_4 meet in $\alpha_1, \alpha_2, \alpha_3$, in c_4 the proper image of c_1', and in a rational curve ρ_9 of order 9, having a triple point at each P_i, and meeting each α_i in 4 points.

In (x'), L_4' does not contain ρ_3', but meets it in 6 points of contact. The image of L_4' is K_6. The jacobian of the web ϕ_i consists of K_6 and of R_2, hence K_6 contains each α_i to multiplicity 2, and each point P_i to multiplicity 3. The surfaces K_6, R_2 intersect in $\alpha_1, \alpha_2, \alpha_3$, each taken twice, and in 6 generators, images of the 6 points of contact of L_4' and ρ_3'.

Among the cubics ϕ_i there is one having a double point at P_i. Moreover, there is a pencil of cubics containing $P_i P_k$. No nodal cubics are included in this pencil. The image of the line $P_i P_k$ in (x') is a straight line. The complete image of this line in (x) consists of $P_i P_k$ and of a residual c_5, hence the line in (x') is a bitangent of L_4'. The line $P_i P_k$ meets R_2 in two points, through each of which passes a generator meeting $\alpha_1, \alpha_2, \alpha_3$. Hence these lines lie on every cubic of the pencil containing $P_i P_k$. The proper residual is a space cubic curve passing through the two remaining basis points and meeting each line α_i twice. Since $P_i P_k$ meets K_6 in two points, the cubic also passes through these points.

The image plane of a nodal cubic (node at P_i) touches L_4' along a conic, since its images in (x) consist of P_i and the cubic surface. Any line in the plane is a bitangent of L_4'. The images of the line consist of P_i and of a rational sextic curve c_6 having a node at P_i. The line of intersection of the image planes of two nodal cubics contains two double points on L_4'.

The image in (x) consists of the two points and of two fundamental cubics, each passing through both basis points, and one remaining basis point; the lines α_i are bisecants of the cubic curves. The four nodal cubics have for images the planes of the tetrahedron having the double points P_i' for vertices. Hence through each image cubic curve pass three nodal cubic surfaces. Since P_i' is on L_4', the four cubic curves p_3 are all on K_6.

The nodal cubics ψ_i also contain the lines g_{ik}. Their image points G_{ik}' are double points on L_4' and lie on every s_6' of the system.

These results are expressed by the following Table.

$$s_1' \sim s_3 : \sum \alpha_i + \sum P_i ,$$

$$c_1' \sim c_6 ,\ p = 1; \qquad [c_6 , \alpha_i] = 4 , \qquad [c_6 , P_i] = 1 ,$$

$$\alpha \sim A_4' : \rho_3' , \quad P_i \sim \pi_i' , \quad R_2 : \sum \alpha \sim \rho_3' , \quad s_1 \sim s_6' : \rho_3' , \quad c_1 \sim c_3' ; \quad [c_3' , \rho_3'] = 2 ,$$

$$K_6 : \sum \alpha_i^2 + \sum P_i^3 + 12 g_i + 4 p_3 \sim L_4' : 12 G'^2 + 4 P'^2 .$$

7. **The involution I.** We may now write at once

$$s_1 \sim s_{15} : \sum \alpha_i^5 + \sum P_i^3 + 12 g + 4 p_3^3 ,$$

$$\alpha_i \sim A_{10} : \alpha_i^4 + 2\alpha^3 + \sum P_i^4 , \qquad R_2 : \sum \alpha_i \sim \rho_9 .$$

The planes $\alpha_i P_k$ contains $g_{k,\, il}$ and $g_{k,\, lk}$. The image of this plane is $A_{10,\, i}$, ψ_k and a proper quadric through the other two fundamental lines and the other fundamental points. The plane and the quadric together form a cubic surface of the web. The image plane in (x') touches L_4' along a conic; the conic contains 5 points G_i' and one point P_i', all double on L_4'. Since each nodal cubic contains three lines g_i it follows that the image plane

contains three points G'_i; it also contains three points P'_i. Hence the surface L'_4 has 16 double points and 16 singular tangent planes; six double points lie in each singular tangent plane and six singular planes pass through each double point. Thus, L'_4 is the Kummer surface. This involution cannot be reduced to that defined by a web of quadrics through six points (Art. 5, Type A). The jacobian of I is made up of the three surfaces A_{10}, the quadric R_2, and the four nodal cubics having nodes at the basis points, each taken twice.

8. **Case B.** Let the rational quartic curve be β_4, and the three basis points be P_1, P_2, P_3. We then have

$$s'_1 \sim s_3 : \beta_4 + \sum P_i,$$

$$c_1 \sim c_5,\ p = 1; \qquad [c_5, \beta_4] = 10, \qquad [c_5, P_i] = 1,$$

$$\beta_4 \sim B'_{10} : \rho'^3, \qquad R_2 : \beta_4 \sim \rho'_2, \qquad s_1 \sim s'_5 : \rho'_2 + 9G'.$$

Through each point P_i can be drawn three lines g meeting β_4 in two points. The images of these lines are points G'. The three nodal cubic surfaces (having nodes at P_i) have a common cubic curve p_3 whose image is a point P'', common to the three singular planes, images of the nodal cubics. The residual intersections of the nodal cubics are three fundamental conics h_2 whose images are points H'.

There is a $(1, 2)$ correspondence of the Geiser type between the plane of ρ'_2 and of $P_1 P_2 P_3$; the fundamental points are P_i and the four points on β_4.

The residual image of ρ'_2 is a plane quartic curve ρ_4 having a double point at each point P_i, meeting β_4 in four points, and meeting R_2 in four other points, all on K_6; through each passes a fundamental line t.

In the involution I the results are

$$s_1 \sim s_{12} : \beta_4^4 + \sum P_i^5 + 9g_i + (R, P) + 4t + 3h_2^2 + p_3^3,$$

$$\beta_4 \sim B_{24} : \beta_4^9 + \sum P_i^{10}, \qquad \rho_4 \sim R_2 : \beta_4,$$

$$(R, P) \equiv \text{plane section of } R_2 \text{ by } P_1 P_2 P_3.$$

In every case the image of a basis point is the surface of the web having that point for node. This will be understood in all subsequent cases.

The surface L'_4 has 13 double points and three singular tangent planes. It is the complete focal surface of a line congruence of order 2 and class 5.*

* Kummer, *Ueber die algebraischen Strahlensysteme, in's Besondere über die der ersten und zweiten Ordnung*, Abhandlungen der k. Akademie der Wissenschaften zu Berlin, 1866, see pp. 88–94.

9. **Case C.** Let the quintic curve be β_5 and the basis points P_i. Then we may write

$$s_1' \sim s_3 : \beta_5 + 2P_i,$$

$$c_1' \sim c_4, \ p = 1; \qquad [c_4, \beta_5] = 8, \qquad [c_4, P_i] = 1,$$

$$\beta_5 \sim B_8' + \rho_1'^3, \quad R_2 : \beta_5 \sim \rho_1', \quad s_1 \sim s_4' : \rho_1' + 8G'.$$

The nodal cubics intersect in the two fundamental conics p_2. The residual image of ρ_1' is $\rho_1 = P_1 P_2$. This line meets R_2 in two points through each of which passes a line t lying on R_2 and K_6. The images of these lines in (x') are the points of contact of ρ_1' and L_4'.

In I we have

$$s_1 \sim s_9 : \beta_5^3 + 2P_i^4 + 8g + 2p_2^2 + 2t + \rho_1,$$

$$\beta_5 \sim B_{18} : \beta_5^5 + 2P_i^8, \qquad \rho_1 \sim R_2 : \beta_5.$$

The surface L_4' has 10 double points and 2 singular planes.

10. **Case D.** Given any point A in (x). The quadric through β_6 and any plane through AP form a composite cubic of the web. The line AP meets a proper cubic of the web through A in a third point B. The points A, B, conjugate in the involution I, are therefore always collinear with P, hence this involution is of the monoidal type, and will not be considered further.

3. Web of Quartics

11. **Simple basis curves.** If the basis curve is simple, the web has no basis points. From the preceding formulas the only condition to be satisfied is

$$4m - p = 30.$$

If β_m is not composite, $m = 11, 10, 9$, or 8. The first of these cases was discussed as Type I in our previous paper.* The others will be designated by A, B, C, and discussed in turn, thus:

$$A. \quad m = 10, \quad p = 10, \qquad B. \quad m = 9, \quad p = 6, \qquad C. \quad m = 8, \quad p = 2.$$

12. **Case A.** The image of a plane s_1' being a surface s_4 through β_{10}, it follows that the image of a line c_1' is a sextic c_6 of genus 2 meeting β_{10} in 22 points. A curve of order m and genus p has

$$\frac{((m-1)(m-2)-2p)}{6}((m-1)(m-2)-2p-8m+22) \tag{9}$$
$$-\frac{m}{24}(m-2)(m-3)(m-13)$$

* L. c.

quadrisecants.* Hence the curve β_{10} has $x = 31$ quadrisecants whose images in (x') are points. The curve may also have conics, cubics $\cdots$ meeting it in $8, 12, 16 \cdots$ points whose images are also points. Let the numbers of these be $y, z, u \cdots$ respectively. The image of s_1 in I is s_{23} having β_{10} 6-fold. Since L' is of order 6, therefore K is of order 12 with β_{10} triple. The plane s_1 meets its image s_{23} in (s_1, K_{12}) and a residual δ_{11} having the points (s_1, β_{10}) triple. The image of δ_{11} in (x') is the double curve δ' of s_7'. A fundamental curve of order k meets s_1 in k points which are simple on (s_1, K_{12}) and $k-1$-fold on δ_{11}.

The genus of δ_{11} is therefore $15 - 3z - 12u - 30v - 60w$. The surface s_6' being rational it follows that the postulation of δ_7' for the adjoint quadrics of s_6' is 10. To a fundamental curve of order k corresponds a $(k(k-1)/2)$-fold point of δ_7', hence by formula (7) we have

$$10 = 14 - p_{\delta_7'} + 1 - 2z - 8u - 20v - 40w.$$

The number of branch points in the $(1, 2)$ correspondence between the points of δ_7' and δ_{11} is therefore, by the Zeuthen formula,

$$\begin{aligned}\eta &= (30 - 6z - 24u - 60v - 120w - 2) \\ &\qquad - 2(10 - 4z - 16u - 40v - 80w - 2) \\ &= 2z + 8u + 20v + 40w + 12.\end{aligned}$$

The 132 intersections of δ_{11} and (s_1, K_{12}) are made up of 90 on β_{10}, of a simple intersection at two points for each of the y conics, a double point on δ_{11} simple on K_{12}, at three points for each of the z cubics, etc., and η other intersections, hence

$$2y + 6z + 12u + 20v + 30w + \eta = 42.$$

The intersection of s_{23} and K_{12} consists of (s_1, K_{12}), of β_{10} taken 6-fold on s_{23}, 3-fold on K_{12}, and of fundamental lines, conics, etc., hence

$$x + 4y + 9z + 16u + 25v + 36w = 84.$$

Similarly, considering the intersection of two surfaces s_{23} of the web, we find

$$x + 8y + 27z + 64u + 125v + 216w = 146.$$

From these equations we have the only possible solution

$$x = 31, \qquad y = 11, \qquad z = 1, \qquad \eta = 14.$$

We have therefore for the correspondence

$$s_1' \sim s_4 : \beta_{10}, \qquad c_1' \sim c_6, \ p = 2, \qquad [c_6, \beta_{10}] = 22,$$

$$s_1 \sim s_6' : 31G' + 11P'^2 + Q'^3,$$

$$K_{12} : \beta_{10} \sim L_6' : 31G'^2 + 11P'^2 + Q'^2, \qquad \beta_{10} \sim B_{22}'.$$

* See Pascal's *Repertorium*, 1st edition, vol. 2, p. 231.

In the involution I we have

$$s_1 \sim s_{23} : \beta_{10}^6 + 31g + 11p_2^2 + q_3^3, \qquad \beta_{10} \sim B_{88} : \beta_{10}^{23}.$$

Each s_4 of the web is a Fano surface.*

The effect of the involution on curves belonging to a surface of the web may be very easily obtained from the results just given; they lead at once to the theorems given by Severi.†

13. **Case B.** We first find $x = 30$ by formula (9), then, proceeding as in Case A, we obtain $y = 12$, $z = 5$, hence

$$s_1' \sim s_4 : \beta_9, \qquad c_1' \sim c_7,\ p = 2; \qquad [c_7, \beta_9] = 26, \qquad s_1 \sim s_7', \qquad \beta_9 \sim B_{26}',$$

$$K_{12} : \beta_9^3 + 30g + 12p_2 + 5q_3 \sim L_6' : 30G' + 12P'^2 + 5Q'^3.$$

In I we have

$$s_1 \sim s_{27} : \beta_9^7 + 30g + 12p_2^2 + 5q_3^3, \qquad \beta_9 \sim B_{104} : \beta_9^{29}.$$

14. **Case C.** Here the characteristics are

$$s_1' \sim s_4 : \beta_8, \qquad c_1' \sim c_8,\ p = 2; \qquad [c_8, \beta_8] = 30,$$

$$\beta_8 \sim B_{30}', \qquad s_1 \sim s_8', \qquad K_{12} : \beta_8^3 \sim L_6'.$$

In I we have

$$s_1 \sim s_{31} : \beta_8^8, \qquad \beta_8 \sim B_{120} : \beta_8^{31},$$

$$x = 31, \qquad y = 10, \qquad z = 9, \qquad u = 1.$$

Each surface of the web of quartics is invariant under I. Any two surfaces of the web meet in β_8 and in a curve c_8 with similar characteristics. There exists a web having c_8 as basis curve, every surface of which is invariant under a second involution I'. Every surface of the pencil of surfaces containing both β_8 and c_8 is invariant under both I and I', but not point for point. In fact the transformation II' is non-periodic for points on a given surface F of the pencil, as we proceed to prove by the method of Severi.‡ Let $|C_4|$ denote the system of plane sections of F. Then

$$\beta_8 = 4C_4 - C_8. \qquad [C_4, C_4] = 4, \qquad [C_4, C_8] = 8, \qquad [C_8, C_8] = 2.$$

By the involution I a plane is transformed into a surface of order 31, having β_8 8-fold while $|C_8|$ remains invariant. Since $|C_4|$ and $|C_8|$ constitute a

* Fano, *Sopra alcune superficie del quarto ordine rappresentabili sul piano doppio*, Rendiconti del Reale Istituto Lombardo, series 2, vol. 39 (1906), pp. 1071–1086.

† Severi, *Complementi alla teoria della base per la totalità delle curve di una superficie algebrica*, Rendiconti del Circolo Matematico di Palermo, vol. 30 (1911), pp. 265–288.

‡ L. c.

base on F, the involution I, for curves on F, is completely expressed by

$$C_4 \sim 31C_4 - 8(4C_4 - C_8) = 8C_8 - C_4, \qquad C_8 \sim C_8.$$

Similarly, I' is expressed by

$$C_4 \sim 8\beta_8 - C_4, \qquad \beta_8 \sim \beta_8,$$

that is, by

$$C_4 \sim 31C_4 - 8C_8, \qquad C_8 \sim 120C_4 - 31C_8.$$

Hence, II' is expressed by

$$C_4 \sim 929C_4 - 240C_8, \qquad C_8 \sim 120C_4 - 31C_8,$$

which is non-periodic. We have therefore proved the following

THEOREM. *A quartic surface through a general curve of order* 8 *and genus* 2 *is invariant under a discontinuous non-periodic group of birational transformations.*

15. **Case D.** In this case $m = 7$ and $p = -2$. We find

$$s_1' \sim s_4 : \beta_7, \qquad c_1' \sim c_9,\ p = 2; \qquad [c_9, \beta_7] = 34,$$

$$s_1 \sim s_9', \qquad K_{12} : \beta_7 \sim L_6'.$$

In the involution I the relation is

$$s_1 \sim s_{35} : \beta_7^9.$$

From the intersection of K_{12} and s_{35} we obtain the equation

$$x + 4y + 9z + 16u + 25v + 36w = 219$$

and from the intersection of two surfaces s_{35},

$$x + 8y + 27z + 64u + 125v + 216w = 623.$$

The postulation for s_{35} gives

$$y + 4z + 10u + 20v + 35w = 88.$$

Two cases are possible. When β_7 consists of a rational quintic β_5 and two lines, α, α_1, we find $x = 31$, made up as follows: first, β_5 has one quadrisecant, by formula (9); the surface of trisecants of β_5 is of order 8, hence 8 lines meet either line and meet β_5 in three points; finally, the congruence determined by the two lines α has 16 lines in common with that of the bisecants of β_5. These results can also be found by other formulas.* The only possible solution for the remaining unknowns is $y = 8$, $z = 10$, $u = 4$, $v = w = 0$. Here $\alpha \sim A_{24} : \beta_5^6 + \alpha_1^6 + \alpha^7$, $\alpha_1 \sim A_{24} : \beta_5^6 + \alpha_1^7 + \alpha^6$, and $\beta_5 \sim B_{88} : \beta_7^{23} + 2\alpha^{22}$.

* See Pascal, l. c., page 231.

In the second case β_7 consists of two space cubics and a line. Here $x = 34$, consisting of 12 bisecants of one cubic meeting the line and the other, and of 10 bisecants of both cubics. The only solution now is

$$y = 4, \quad z = 16, \quad u = 0, \quad v = 1, \quad w = 0.$$

For the line we have

$$\alpha \sim A_{24} : \alpha^7 + 2\beta_3^6,$$

and for each cubic

$$\beta_3 \sim B_{56} : \alpha^{14} + \beta_3^{15} + \beta_3^{14}.$$

16. **Double basis curve.** When the web of surfaces of the $(1, 2)$ correspondence are of order n and have a common double curve of order m and genus p, the equivalence is $(12n - 32)m - 16p + 16$ and the postulation is $(3n - 4)m - 5p + 5$. If there is also a simple basis curve of order m' and genus p' meeting the double curve in s points, the additional equivalence is $(3n - 4)m' - 2p' + 2 - 5s$ and the additional postulation is $nm' - p' + 1 - 2s$.

17. **Quartics with a double line α.** In this case the preceding formulas become

$$8m' - 2p' + 2 - 5s + \xi = 30,$$

$$4m' - p' + 1 - 2s + \xi = 18,$$

hence $\xi + s = 6$ and $4m' - p' = 11 + 3s$, of which the possible solutions are

	ξ	s	m'	p'
A	4	2	4	-1
B	3	3	5	0
C	2	4	6	1
D	1	5	7	2
E	0	6	8	3
F	0	6	7	-1

18. **Case A.** The simple basis quartic consists of two conics β_2, $\bar{\beta}_2$, each meeting α once. Hence

$$s_1' \sim s_4 : \alpha^2 + \beta_2 + \bar{\beta}_2 + 4P,$$

$$c_1' \sim c_8, \ p = 1; \quad [\alpha, c_8] = 6, \quad [\beta_2, c_8] = 7, \quad [\beta_2, c_8] = 7,$$

so that $\alpha \sim A_6'$; $\beta_2 \sim B_7'$; $\bar{\beta}_2 \sim \bar{B}_7'$. The lines meeting α, β_2, and $\bar{\beta}_2$ lie on a ruled quartic R_4 having α triple, which meets a quartic surface of the web in 6 generators whose images are coplanar points on the sextic curve ρ_6', image of R_4. The image of a basis point P is a singular tangent plane π' of L_4'. The complete image in (x) of π' is the point P and the quartic of the web which has a node at P. There are 12 other singular tangent planes of L_4' corresponding to the composite quartics of the web, one of whose components is either: (1) the plane of the conic β_2 or $\bar{\beta}_2$, (2) the plane of α and a

point P, (3) the quadric through α, β_2 or $\overline{\beta}_2$ and two of the points P. There are also 16 fundamental curves in (x) whose images are points double on L_4', namely, 8 lines through the points P, meeting α and β_2 or $\overline{\beta}_2$, the line of intersection of the planes of β_2, $\overline{\beta}_2$, a quintic curve common to the four nodal quartics and 6 cubic curves, each lying on 3 of the nodal quartics. The existence of the last seven curves follows from the fact that since two quartics of the web meet in a c_8 of genus 1, two nodal quartics must meet in a composite c_8 consisting of a quintic through the points P meeting α, β_2, and $\overline{\beta}_2$ each in 4 points, and a cubic through 2 nodes meeting α in 2 points and each conic in 3 points. It is readily verified that in each singular tangent plane of L_4' are 6 double points. The surface L_4' is thus the Kummer surface.

The image of L_4' is $K_8 : \alpha^4 + \beta_2^2 + \overline{\beta}_2^2$ meeting R_4 in 12 generators r, images of the 12 contacts of ρ_6' with L_4'. The residual image in (x) of ρ_6' is a curve ρ_{29}. This follows from the fact that the image of s_1 is s_8' whose residual image in (x) is $s_{27} : \alpha^{13} + \beta_2^7 + \overline{\beta}_2^7 + 12r$, meeting R_4 in 29 generators whose images in I are the 29 points in which s_1 meets ρ_{29}. A line meeting ρ_6' has for image a c_7 meeting α in 5 points and each conic β_2 in 6 points, hence

$$\alpha \sim A_6' : \rho_6'; \qquad \beta_2 \sim B_7' : \rho_6'; \qquad \overline{\beta}_2 \sim \overline{B}_7' : \rho_6'.$$

The characteristics of the involution I are therefore

$$s_1 \sim s_{27} : \alpha^{13} + \beta_2^7 + \overline{\beta}_2^7 + 4P^8 + 12r + \rho_{29},$$

$$\alpha \sim A_{20} : \alpha^{10} + \beta_2^5 + \overline{\beta}_2^5, \qquad \beta_2 \sim B_{24} : \alpha^{11} + \beta_2^7 + \overline{\beta}_2^6,$$

$$\overline{\beta}_2 \sim \overline{B}_{24} : \alpha^{11} + \beta_2^6 + \overline{\beta}_2^7, \qquad \rho_{29} \sim R_4 : \alpha^3 + \beta_2 + \overline{\beta}_2.$$

This transformation cannot be reduced to either of the previous ones in which K is equivalent to the Kummer surface.

19. **Case B.** The simple basis curve is now a rational quintic β_5, meeting α in 3 points. The web has three isolated basis points P.

$$s_1' \sim s_4 : \alpha^2 + \beta_5 + 3P,$$

$$c_1' \sim c_7,\ p = 1; \qquad [\alpha, c_7] = 5, \qquad [\beta_5, c_7] = 13,$$

so that

$$\alpha \sim A_5', \qquad \beta_5 \sim B_{13}'.$$

The bisecants of β_5 which meet α lie on a ruled quartic R_4 having α triple and meeting a quartic of the web in 5 generators whose images are coplanar points on the quintic image curve ρ_5' of R_4. The image of a basis point P is a singular tangent plane π' of L_4', whose residual image in (x) is the quartic ψ of the web having a node at P. There are 3 other singular tangent planes of L_4' corresponding to the 3 composite quartics of the web; one of the components is the plane π through α and a point P. There are 6 lines g, two

through each point P, meeting α and β_5. The curve β_5 has a four-fold secant d, having a point D' for image in (x'). Two composite quartics meet in a conic γ_{ik}, having a point Γ'_{ik} for image. Thus

$$[\alpha, \gamma_{12}] = 1, \qquad [\beta_5, \gamma_{12}] = 5, \qquad [P_3, \gamma_{12}] = 1.$$

A composite quartic meets a nodal quartic in a cubic β_{ik} having a point B'_{ik} for image.

$$[\alpha, \beta_{23}] = 2, \qquad [\beta_5, \beta_{23}] = 6, \qquad [P_2, \beta_{23}] = [P_3, \beta_{23}] = 1.$$

Finally, all the nodal quartics pass through a common quartic δ

$$[\alpha, \delta] = 3, \qquad [\beta_5, \delta] = 7, \qquad [P, \delta] = 1.$$

The relations of the lines and surfaces are represented by the following

Table

	g_{11}	g_{21}	g_{31}	g_{12}	g_{22}	g_{32}	γ_{12}	γ_{13}	γ_{23}	β_{23}	β_{13}	β_{12}	δ	d
$f_1\ \pi_1$	*			*			*	*		*				*
$f_2\ \pi_2$		*			*		*		*		*			*
$f_3\ \pi_3$			*			*		*	*			*		*
ψ_1	*			*					*		*	*	*	
ψ_2		*			*			*		*		*	*	
ψ_3			*			*	*			*	*		*	

The surface L'_4 has 14 double points and 6 singular planes. In each singular plane lie 6 double points, lying on a conic. Through 6 of the double points pass 2 singular planes, and through the remaining 8 pass 3. The surface is that discussed by Kummer.*

The image of L'_4 is $K_8 : \alpha^4 + \beta_5^2$, meeting R_4 in 10 generators r, images of the 10 contacts of ρ'_5 with L'_4. The residual image of R_4 is ρ_{19}. This follows from the fact that the image of s_1 is $s'_7 : \rho'_5$ whose image in (x) is $s_{23} : \alpha^{11} + \beta_5^6 + 10r$; it meets R_4 in 19 generators, images of the 19 points in which s_1 meets ρ_{19}. Since a line meeting ρ'_5 has for image a sextic meeting α in 4 and β_5 in 11 points, we see that $\alpha \sim A'_5 : \rho'_5$ and $\beta_5 \sim B'_{13} : \rho_5'^2$.

In I we have

$$s_1 \sim s_{23} : \alpha^{11} + \beta_5^6 + 10r + \rho_{19}, \qquad \alpha \sim A_{16} : \alpha^8 + \beta_5^4,$$

$$\beta_5 \sim B_{44} : \alpha^{20} + {}^{12}_{5}, \qquad P \sim \psi : \alpha^2 + \beta_5 + P^2, \qquad \rho_{19} \sim R_4 : \alpha^3 + \beta_5,$$

$$x = 7, \qquad y = 3, \qquad z = 3, \qquad u = 1.$$

* L. c., p. 87.

20. **Case C.** In the web are two composite quartics and two nodal ones. There are four lines g. The curve β_6 has 2 four-fold secants d_1, d_2 apart from α; they have points D_1', D_2' for images. The 2 nodal quartics intersect in 2 space cubics γ having the characteristics

$$[\alpha, \gamma] = 2, \quad [\beta_6, \gamma] = 6, \quad [P, \gamma] = 1.$$

The image of each is a point in (x').

The nodal quartics intersect the composite quartics in a pair of conics conjugate in I and in the quartic curve δ_4 having the scheme

$$[\alpha, \delta_4] = 2, \quad [\beta_6, \delta_4] = 10, \quad [P, \delta_4] = 1.$$

The surface L_4' has 12 singular points and four singular planes; 6 double points lie in each singular plane. It is not a focal surface of any line congruence of order two. We may now write

$$s_1' \sim s_4 : \alpha^2 + \beta_6 + 2P, \quad [\alpha, \beta_6] = 4,$$
$$c_1' \sim c_6; \quad p = 1; \quad [c_6, \alpha] = 4, \quad [c_6, \beta_6] = 12,$$
$$R_4 : \alpha^3 + \beta_6 \sim \rho_4'; \quad \alpha \sim A_4' : \rho_4'; \quad \beta_6 \sim B_{12}' : \rho_4'^2,$$
$$s_1 \sim s_6' : \rho_4', \quad K_8 : \alpha^4 + \beta_6^2 + 8r \sim L_4' : 12P'^2.$$

In the involution I we have

$$s_1 \sim s_{19} : \alpha^9 + \beta_6^5 + 8r + \rho_{11}, \quad \alpha \sim A_{12} : \alpha^6 + \beta_6^3,$$
$$\beta_6 \sim B_{40} : \alpha^{18} + \beta_6^{11}, \quad \rho_{11} \sim R_4 : \alpha^3 + \beta_6, \quad P \sim \psi,$$
$$x = 6, \quad y = 4, \quad z = 2.$$

21. **Case D.** We may write at once

$$s_1' \sim s_4 : \alpha^2 + \beta_7 + P, \quad [\alpha, \beta_7] = 5,$$
$$c_1' \sim c_5, \; p = 1; \quad [c_5, \alpha] = 3, \quad [c_5, \beta_7] = 11, \quad [c_5, P] = 1,$$
$$R_4 : \alpha^3 + \beta_7 \sim \rho_3'; \quad \alpha \sim A_3' : \rho_3'; \quad \beta_7 \sim B_{11}' : \rho_3'^2,$$
$$s_1 \sim s_5' : \rho_3'; \quad K_8 : \alpha^4 + \beta_7^2 \sim L_4' : 10P'^2.$$

In the involution I

$$s_1 \sim s_{15} : \alpha^7 + \beta_7^4 + 6r + \rho_5,$$
$$\alpha \sim A_8 : \alpha^4 + \beta_7^2; \quad \beta_7 \sim B_{36} : \alpha^{15} + \beta^{10}; \quad \rho_5 \sim R_4 : \alpha^3 + \beta_7; \quad P \sim \psi_4,$$

$x = 6$, $y = 4$. The lines are 2 through P and 4 quadrisecants of β_7.

22. **Case E.** Here we have $s_1' \sim s_4 : \alpha^2 + \beta_8$, $[\alpha, \beta_8] = 6$.

$$c_1' \sim c_4,\ p = 1; \quad [c_4, \alpha] = 2, \quad [c_4, \beta_8] = 10,$$

$$R_4 : \alpha^3 + \beta_8 \sim \rho_2'; \quad \alpha \sim A_2' : \rho_2'; \quad \beta_8 \sim B_8' : \rho_2'^2,$$

$$s_1 \sim s_4' : \rho_2'; \quad K_8 : \alpha^4 + \beta_8^2 \sim L_4' : 8D'^2.$$

In I we have

$$s_1 \sim s_{11} : \alpha^5 + \beta_8^3 + 8d + 4r + \rho_1,$$

$$\alpha \sim A_4 : \alpha^2 + \beta_8 + 8d; \quad \beta_8 \sim B_{32} : \alpha^{14} + \beta_8^9; \quad \rho_1 \sim R_4 : \alpha^3 + \beta_8.$$

23. **Case F.** In case $m = 7$, $p = -1$, then $s = 6$ and $\xi = 0$. The basis curve consists of a rational sextic β_6 which meets α in 5 points, and a line $\bar{\alpha}$ meeting α but not meeting β_6. Through β_6 pass ∞^1 cubic surfaces, the residual intersection consisting of the four-fold secant d of β_6, and α counted twice.*

In the pencil of cubic surfaces is a ruled cubic R_3 having α for double directrix, and d for simple directrix. Let $\bar{\alpha}$ be defined by $x_1 = 0$, $x_2 = 0$, and α by $x_1 = 0$, $x_3 = 0$. Then if $F = 0$ is any cubic of the pencil and $S = 0$ any quartic of the web, we may express the correspondence by

$$x_1' = x_1 R_3, \quad x_2' = x_2 R_3, \quad x_3' = x_1 F, \quad x_4' = S$$

and

$$s_1' \sim s_4 : \alpha^2 + \bar{\alpha} + \beta_6.$$

Since

$$[\alpha, \beta_6] = 5, \quad [\alpha, \bar{\alpha}] = 1, \quad [\bar{\alpha}, \beta_6] = 0,$$

we have

$$c_1' \sim c_5,\ p = 1; \quad [c_5, \alpha] = 2, \quad [c_5, \bar{\alpha}] = 3, \quad [c_5, \beta_6] = 11,$$

$$x_1 = 0 : \alpha + \bar{\alpha} \sim \rho' (x_1' = 0,\ x_3' = 0),$$

$$R_3 : \alpha^2 + \beta_6 \sim r' (x_1' = 0,\ x_2' = 0), \quad \beta_6 \sim B_{11}' : \rho' + r'^5,$$

$$\alpha \sim x_1' = 0 \text{ counted twice}, \quad \bar{\alpha} \sim \bar{A}_3' : r' + \rho', \quad s_1 \sim s_5' : r'^2 + \rho',$$

$$c_1 \sim c_1',\ p = 0; \quad [c_1', r'] = 3, \quad [c_1', \rho'] = 1.$$

Let $\bar{\alpha}$ meet R_3 in P, apart from α. A plane passed through any generator h of R_3 and P cuts R_3 in h and in a conic. The conic passes through P, through the point (h, α) and through five points of β_6 not on h. Since the image of a point on α is a conic, and of a point on β_6 or $\bar{\alpha}$ is a straight line, it follows that the image of the conic is a point. As h describes R_3 the image point

* Noether, *Zur Grundlegung der Theorie der algebraischen Raumcurven*, Abhandlungen der königlich Preussischen Akademie der Wissenschaften zu Berlin vom Jahre 1882. See page 86. Case a_0'.

of the associated conic describes r'. The image of a plane $x'_1 = \tau x'_2$ through r' consists of R_3 and of the plane $x_1 = \tau x_2$ through $\bar{\alpha}$. The complete image of the latter plane consists of A'_3 and of the plane $x'_1 = \tau x'_2$ taken twice. A straight line in $x'_1 = \tau x'_2$ has for image the intersection of the plane $x_1 = \tau x_2$ and a general quartic s of the web. It consists of the fundamental line $\bar{\alpha}$ and of a general cubic through the point on α and the six points on β_6. Hence the $(1, 2)$ correspondence between the two planes is of the Geiser type, having the points on β_6 and on α for fundamental points. The line in (x') meets r' in a point whose image is a conic, intersection of R_3 and S. The plane $x_1 = \tau x_2$ meets the conic in P and in one other point lying on the nodal cubic curve in which the plane meets R_3. This nodal cubic is the image of the point on α in the involution I. As τ varies, the nodal cubic describes R_3, hence we see that the image of R_3 in I is the point $(\alpha, \bar{\alpha})$.

Let g_1, g_2 be the two generators of R_3 through $(\alpha, \bar{\alpha})$, and let τ_i define the plane of the pencil $x_1 = \tau x_2$ through g_i. The nodal cubic now consists of the conic and g_i. In a general plane the jacobian of the net of cubics, a sextic having double points at all seven basis points, is the partial section of K_8 by the plane. In the plane τ_i the conic is part of the jacobian, hence the two conics lie on K'_8. Since the image of r' is composite, we conclude that r' is a bitangent of L'_4, the points of tangency being at the image points of the two conics on K_8.

A rational β_6 has a ruled surface of trisecants of order 20. Since α is a five-fold secant, it counts for 10 trisecants, hence $\bar{\alpha}$ meets 10 others. In the planes of the pencil $x_1 = \tau x_2$ these lines lie on the jacobians of the net of cubic curves, hence these 10 lines all lie on K_8.

The plane $x_1 = 0$ meets β_6 in one point H not on α. A line through H in this plane goes into a point of ρ'. The lines HP and d constitute a composite conic on R_3, the image of which is the point $(0, 0, 0, 1)$ in (x'). The two lines of the pencil H which lie on K_8 have for images the points of contact of L'_4 and ρ'. To the points (λ) of any line g through H correspond the elements in the plane $x'_1 = \lambda x'_3$ through G'.

In I we now have

$$s_1 \sim s_{12} : \alpha^5 + \bar{\alpha}^4 + \beta_6^3 .$$

24. **The fundamental line α.** To obtain the images of the points of α, first consider the straight lines meeting it. In the equations $x'_1 = x_1 R_3$, etc., replace x_1 by kx_3, x_2 by μx_4 and divide by x_3^2. Then replace x_3 by 0 to obtain the image of the point $(0, \mu, 0, 1)$. The resulting equations have the form $x'_1 = 0$, $x'_2 = \mu(k, \mu)$, $x'_3 = k(k, \mu)$, $x'_4 = (k, \mu)$, all the second members being non-homogeneous polynomials of order 2 in k and in μ. By eliminating k, we have a conic containing the parameter μ to degree 8, of

which 6 roots (intersections with β_6 and with $\bar{\alpha}$) are constant, hence there results a quadratic system of conics

$$\mu^2 C' + \mu \bar{C}' + x_2' u' = 0,$$

having the section of L_4' by the plane $x_1' = 0$ for envelope. Similarly, by eliminating μ we obtain the quadratic system

$$k^2 C_1' + k\bar{C}_1' + x_3' u' = 0.$$

Every direction in the pencil determined by μ and k has the same image point in (x').

By regarding μ, k as non-homogeneous point coördinates in an auxiliary plane π, the preceding equations define a $(1, 2)$ correspondence of the Geiser type between $x_1' = 0$ and π.

Now let a plane s_1 meet α in μ_1 and another plane s_2 meet α in μ_2. The conics $c_2'(\mu_1)$, $c_2'(\mu_2)$ meet in four points, each of which has two images (μ, k) on α. Let them be (μ_1, k_{11}), (μ_1, k_{21}), (μ_1, k_{31}), (μ_1, k_{41}) and (μ_2, k_{12}), (μ_2, k_{22}), (μ_2, k_{32}), (μ_2, k_{42}). These directions define 4 tangent planes to the image s_{12} of s_2 at μ_1 and of the image s_{12} of s_1 at μ_2 respectively.

To obtain the tangent of the fifth branch of any s_{12} at μ, consider the section of s_1 with its own image s_{12}. It consists of (s_1, K_8) and a residual c_4 in $(2, 1)$ correspondence with the points of the double cubic curve δ' of s_5'. The double cubic meets r' in the points of contact of L_4' and r', images of the two conics (R_3, K_8) and meets ρ' in a variable point M_2'.

The curve c_4 has a double point at T, where s_1 meets $\bar{\alpha}$. We have seen that every plane $x_1 = \tau x_2$ through $\bar{\alpha}$ is invariant in I. It meets s_1 in a line whose image is a curve of order 8, meeting it in 6 points on K_8 and in a pair of conjugate points in I. As τ varies, these conjugate points describe c_4. When $\tau = 0$, the points are μ and a definite point M. The line MH is on s_{12}, and forms, with fundamental lines, the complete intersection $x_1 = 0$, s_{12}.

The conic $c_2'(\mu)$ meets $x_3' = 0$ $(k = 0)$ in two points $M_1' = (\mu, 0)$ and $(\bar{\mu}, k)$ and $M_2' = (\mu, k_1)$ and $(\mu_1, 0)$. The double curve δ' passes through M_2' and its tangent fixes λ. The direction of the tangent to c_4 at μ is in the plane k_1.

Now consider any point D in $x_1 = 0$. Draw the line DH, and call the point in which it meets α by its parameter μ. The conic $c_2'(\mu)$ defined by this point meets ρ' in D_1' and D_2'. The value of k defined by D_2' is therefore independent of λ on the line DH. Any plane s_1 meets $x_1 = 0$ in a line having a point on every line g through H. The image of this point is a definite element of α, hence the following

THEOREM. *All the surfaces s_{12} have a common tangent plane at every point of α.*

The table for I may now be written as follows:

$s_1 \sim s_{12} : \alpha^5$ (one branch fixed) $+ \bar{\alpha}^4 + \beta_6^3$, $\quad \alpha \sim (x_1 = 0$ taken twice),

$\bar{\alpha} \sim \bar{A}_8 : \alpha^3 + \bar{\alpha}^3 + \beta_6^2, \quad \beta_6 \sim B_{28} : \alpha^{11} + \bar{\alpha}^{10} + \beta_6^7, \quad (\alpha, \bar{\alpha}) \sim R_3,$

$x = 10 +$ two lines $x_1 = 0$ on K_8, $\quad y = 1 +$ two conics of R_3 on K_8.

4. Web of quintics

25. **Simple basis curve impossible.** For a simple basis curve of order m and genus p we require

$$11m - 2p + 2 + \xi = 123, \qquad 5m - p + 1 + \xi = 52.$$

Hence $m - \xi = 19$, $6m - p = 70$. If $\xi = 0$, then $m = 19$, so that the residual intersection of two quintics of the web is a sextic curve of genus five, which is impossible; if $\xi = 1$, the residual is a quintic of genus five, which is also impossible, and so on.

26. **Double basis line α and simple basis curve β of order m' and genus p'.** Here we have

$$11m' - 2p' + 2 - 5s + \xi = 79, \qquad 5m' - p' + 1 - 2s + \xi = 36,$$

so that $p' + 3s = 6m' - 42$, and $s + \xi = m' - 7$.

The variable curve of intersection of two surfaces of the web is of genus 3. Pass a plane through α. It meets the surfaces of the web in cubic curves, any two of which meet in 9 points of which $m - s$ are on β_m, and therefore $9 - m + s$ on the variable curve of intersection. Thus $m - s \leqq 7$, and consequently $\xi = 0$. The variable curve meets α in $12 - s$ points, and meets β_{7+s} in $44 - 3s$ points.

A surface Δ_4 contains α and passes through β_{7+s}. The image of Δ_4 is a straight line δ' in (x'). We may now write

$s_1' \sim s_5 : \alpha^2 + \beta_{7+s}; \quad [\alpha, \beta_{7+s}] = s; \quad p$ of $\beta_{7+s} = 3s,$

$c_1' \sim c_{14-s}, \quad p = 3; \quad [\alpha, c_{14-s}] = 12 - s, \quad [\beta_{7+s}, c_{14-s}] = 44 - 3s,$

$\Delta_4 : \alpha + \beta_{7+s} \sim \delta', \quad s_1 \sim s_{14-s}' : \delta'^{11-s}, \quad \alpha \sim A_{12-s}' : \rho'^{9-s};$

$\beta_{7+s} \sim B_{44-3s}' : \delta'^{37-3s}, \quad K_{12} : \alpha^6 + \beta_{7+s}^2 \sim L_8' : \delta'^4,$

and in the involution I

$$s_1 \sim s_{25-s} : \alpha^{17-s} + \beta_{7+s}^3,$$

$$\alpha \sim A_{44-s} : \alpha^{16-s} + \beta_{7+s}^3; \qquad \beta_{7+s} \sim B_{67-3s} : \alpha^{46-3s} + \beta_{7+s}^8.$$

The range of variation of s is $0 \leqq s \leqq 8$, hence there are nine distinct cases of involutions defined by quintics having a double line.*

27. **Quintics with a double conic.** From Article 16 we have

$$p' + 3s = 6m' - 25, \qquad s + \xi = m' - 1.$$

The only possible solutions are $\xi = 0$, and $7 \leqq m' \leqq 12$. When $m' = 7$, β_7 is composite and two types appear, each having $p' = -1$. In the first, $\beta_7 = \beta_6 + \alpha$, the sextic being rational and not meeting the line α. In the second case, $\beta_7 = \beta_4 + \beta_3$, both rational. We have for all these cases (writing m instead of m')

$$s_1' \sim s_5 : \gamma_2^2 + \beta_m; \qquad [\beta_m, \gamma_2] = m - 1, \qquad p \text{ of } \beta_m \text{ is } 3m - 22,$$

$$c_1' \sim c_{17-m}, \qquad p = 2; \qquad [c_{17-m}, \gamma_2] = 17 - m, \qquad [c_{17-m}, \beta_m] = 49 - 3m,$$

$$\gamma_2 \sim \Gamma'_{17-m} : \rho'^2; \qquad R_1 : \gamma_2 \sim \rho'; \qquad \beta_m \sim B'_{49-3m} : \rho',$$

$$s_1 \sim s^1_{17-m} : \rho'; \qquad K_{15} : \gamma_2^6 + \beta_m^3 + 3r \sim L_6^3.$$

In the involution I we have

$$s_1 \sim s_{83-5m} : \gamma_3^{33-2m} + \beta^{17-m} + 3r + \rho_{14-m},$$

$$\gamma_2 \sim \Gamma_{83-5m} : \gamma_2^{33-2m} + \beta_m^{17-m}; \qquad \beta_m \sim B_{244-15m} : \gamma_2^{97-6m} + \beta_m^{50-3m},$$

$$\rho_{14-m} \sim R_1 : \gamma_2; \qquad x = (m^3 - 24m^2 + 197m - 402)/6.$$

28. **Quintics with two non-intersecting double lines.** The only possible case is that in which the residual basis curve β_m of order m is of genus 4; there are $11 - m$ isolated basis points. The fundamental lines α and $\bar{\alpha}$, β_m lie on a ruled surface R_6 of order 6, genus 4. We now have, for $7 \leqq m \leqq 11$,

$$s_1' \sim s_5 : \alpha^2 + \bar{\alpha}^2 + \beta_m + (11 - m)P; \qquad p \text{ of } \beta_m \text{ is } 4;$$

$$[\alpha, \beta_m] = [\bar{\alpha}, \beta_m] = m - 3,$$

$$c_1' \sim c_{17-m}, \qquad p = 1; \qquad [c_{17-m}, \alpha] = [c_{17-m}, \bar{\alpha}] = 15 - m,$$

$$[c_{17-m}, \beta_m] = 12,$$

$$\alpha \sim A'_{15-m} : \rho'_{18-m}; \qquad \bar{\alpha} \sim \bar{A}_{15-m} : \rho'_{18-m}; \qquad R_6 : \alpha^3 + \bar{\alpha}^3 + \beta_m \sim \rho'_{18-m},$$

$$\beta_m \sim B'_{12} : \rho'_{18-m}; \qquad s_1 \sim s'_{17-m} : \rho'_{18-m},$$

$$K_{10} : \alpha^4 + \bar{\alpha}^4 + \beta_m^2 + (36 - 2m)r + (11 - m)P^2 \sim L_4',$$

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* The webs of quintics having a double line can be generalized immediately to surfaces of order n having α to multiplicity $n - 3$. The general involution is

$$s_1 \sim s_{9n-s-20} : \alpha^{9n-s-28} + \beta^3_{7+s}.$$

No new fundamental elements appear.

and in the involution I

$$s_1 \sim s_{78-5m} : \alpha^{31-2m} + \bar{\alpha}^{31-2m} + \beta^{16-m},$$

$$\alpha \sim A_{69-5m} : \alpha^{28-2m} + \bar{\alpha}^{27-2m} + \beta^{14-m}; \qquad \bar{\alpha} \sim A_{69-5m} : \alpha^{27-2m} + \bar{\alpha}^{28-2m}, \text{ etc.}$$

$$\beta_m \sim B_{54} : \alpha^{21} + \bar{\alpha}^{21} + \beta_m^{12}, \qquad \rho \sim R_6 : \alpha^3 + \bar{\alpha}^3 + \beta_m .$$

The order of ρ is $m^2 - 34m + 282$.

29. **Double space cubic γ_3.** The simple curve $\beta_{m'}$ satisfies the conditions

$$11m' - 2p' + 2 - 5s + \xi = 23,$$

$$5m' - p' + 1 - 2s + \xi = 14,$$

so that $p' + 3s = 6m' - 8$, $s + \xi = m' + 5$. The following cases exist.

	A	B	C	D	E	F	G	H	I	J
m'	3	3	4	4	5	5	6	6	7	7
p'	1	−2	1	−2	1	−2	1	−2	1	−2
ξ	5	4	4	3	3	2	2	1	1	0

30. **Case A.** This case is reducible to Case A of Article 5.

31. **Case B.** The simple cubic consists of two bisecants α, $\bar{\alpha}$ of γ_3, and a line β not meeting γ_3. Hence

$$s_1' \sim s_5 : \gamma_3^2 + \alpha + \bar{\alpha} + \beta + 4P,$$

$$c_1' \sim c_{10}, \quad p = 1; \quad [c_{10}, \gamma_3] = 16, \quad [c_{10}, \alpha] = 2, \quad [c_{10}, \beta] = 8.$$

The bisecants of γ_3 which meet β lie on $R_4 : \gamma_3^2 + \beta$ and have for images in (x') the points of a curve ρ_7'. The quadric H_2 through γ_3, α, $\bar{\alpha}$ and a cubic through γ_3, β, and the points P form a composite quintic of the web. The image of H_2 is therefore a line h' and the complete image of h' consists of H_2 and a quintic curve h_5 common to the cubics of the pencil. The quadric H_2 meets any s_5 of the web in a conic, the partial image of a point of h'. The conic meets γ_3 in 3 points, α and $\bar{\alpha}$ each in one, and β in 2 points. The surface L_4' has 16 singular tangent planes, images of the 8 composite quintics containing a quadric through γ_3, α or $\bar{\alpha}$ and one of the points P, the 4 containing a plane through β and a point P, and the 4 nodal quintics. Two nodal quintics intersect in two quintic curves t_5. Through each point P passes a bisecant g of γ_3 and a conic p_2 meeting β twice, γ_3 three times, and α or $\bar{\alpha}$. There are also two lines d meeting γ_3, α, $\bar{\alpha}$, and β. The four quintics t_5, the 8 lines g, the 8 conics p_2 and the 2 lines d have for images the 16 double points of L_4'.

The line h' is bitangent to L_4', the points of contact being D_i', whose images are conics lying on K_{10} and H_2. The curve ρ_7' is tangent to L_4' at 14 points R', images of 14 generators r in which K_{10} meets R_4.

We may now write

$$\alpha \sim A_2' : h', \quad \overline{\alpha} \sim \overline{A}_2' : h_2', \quad \beta \sim B_8' : \rho_7' + h'^2, \quad \gamma_3 \sim \Gamma_{16}' : \rho_7'^2 + h'^2,$$

$$s_1 \sim s_{10}' : h'^2 + \rho_7', \quad c_1 \sim c_5', \quad p = 0; \quad [c_5', h'] = 2, \quad [c_5', \rho_7'] = 4,$$

$$K_{10} : \gamma_3^4 + \alpha^2 + \overline{\alpha}^2 + \beta^2 \sim L_4'.$$

and in the involution I

$$s_1 \sim s_{41} : \gamma_3^{16} + \alpha^8 + \overline{\alpha}^8 + \beta^9,$$

$$\alpha \sim A_8 : \gamma_3^3 + \alpha^2 + \overline{\alpha} + \beta^2, \quad \overline{\alpha} \sim \overline{A}_8 : \gamma_3^3 + \alpha + \overline{\alpha}^2 + \beta^2,$$

$$\beta \sim B_{32} : \gamma_3^{12} + \alpha^6 + \overline{\alpha}^6 + \beta^8, \quad \gamma_3 \sim \Gamma_{66} : \gamma_3^{26} + \alpha^{13} + \overline{\alpha}^{13} + \beta^{14},$$

$$h_5 \sim H_2 : \gamma_3 + \alpha + \overline{\alpha}, \quad \rho_{45} \sim R_4 : \gamma_3^2 + \beta.$$

Each point P goes into the quintic of the web having a node at P.

32. **Case C.** Here we have a quartic curve β_4, $p = 1$, $[\beta_4, \gamma_3] = 5$, and 4 basis points P, hence we may write

$$s_1' \sim s_5 : \gamma_3^2 + \beta_4 + 4P,$$

$$c_1' \sim c_9, \quad p = 1; \quad [c_9, \gamma_3] = 15, \quad [c_9, \beta_4] = 9.$$

The bisecants of γ_3 which meet β_4 lie on $R_6 : \gamma_3^3 + \beta_4$ and have for images in (x') the points of a curve ρ_8'. The surface L_4' has 10 singular tangent planes, images of the 6 composite quintics of the web, one of whose components is a quadric through γ_3 and 2 of the points P, and the four nodal quintics. Two nodal quintics intersect in a fundamental sextic d_6 meeting γ_3 in 10 points, β_4 in 6, and through the four points P, and a cubic t_{ik} meeting γ_3 in 5 points, β_4 in 3, and through 2 of the points P. Through each point P passes a fundamental conic p_2 meeting γ_3 in 3 points, β_4 in 2 points, and a bisecant g of γ_3. The sextic, the 6 cubics, the 4 conics, and the four lines have for images the 15 double points of L_4'. The surface L_4' is the focal surface of a line congruence of order 2 and class 3. The curve ρ_8' is tangent to L_4' at 16 points R', images of the 16 generators r in which K_{10} meets R_6.

We may now write

$$\gamma_3 \sim \Gamma_{15}' : \rho_8'^2, \quad \beta_4 \sim B_9' : \rho_8',$$

$$s_1 \sim s_9' : \rho_8', \quad c_1 \sim c_5', \quad [c_5', \rho_8'] = 6, \quad K_{10} : \gamma_3^4 + \beta_4^2 \sim L_4'.$$

In I the results are

$$s_1 \sim s_{38} : \gamma_3^{15} + \beta_4^8, \quad \gamma_3 \sim \Gamma_{63} : \gamma_3^{25} + \beta_4^{13},$$

$$\beta_4 \sim B_{39} : \gamma_3^{15} + \beta_4^9, \quad \rho_{45} \sim R_6 : \gamma_3^3 + \beta_4.$$

33. **Case D.** The basis quartic consists of two lines α, $\bar{\alpha}$ and a conic β_2, each a bisecant of γ_3; the system has three basis points P. We have therefore

$$s_1' \sim s_5 : \gamma_3^2 + \alpha + \bar{\alpha} + \beta_2 + 3P,$$

$$c_1' \sim c_9, \qquad p = 1; \qquad [c_9, \gamma_3] = 14, \qquad [c_9, \alpha] = 2, \qquad [c_9, \beta_2] = 8.$$

The quadric H_2 through γ_3, α, and $\bar{\alpha}$ with a cubic of the pencil through γ_3, β_2 and the points P make up a composite quintic of the web. Hence the image of H_2 is a line h' whose image in (x) is H_2 and the quartic h_4 common to the pencil of cubics. There are 10 singular tangent planes to L_4', images of the composite quintics of the web, one of whose components is either a quadric through γ_3 and α or $\bar{\alpha}$ and one point P, or the plane of β_2, or a nodal quintic.

The nodal quintics have a common quintic curve, and meet by pairs in one of three quartic curves. Through each point P passes a bisecant g of γ_3. In the plane of β_2 are two lines t meeting γ_3 and α or $\bar{\alpha}$. There are 6 fundamental conics, two through each point P, meeting γ_3 3 times, α, $\bar{\alpha}$ and β_2 each once.

The quintic curve, the three quartics, the six conics, and the five lines have for images the 15 double points of L_4'.*

The bisecants of γ_3 which meet β_2 lie on a ruled surface R_4 having γ_3 double; they have for images the points of a curve ρ_6' touching L_4' in 12 points R', images of the 12 generators of R_4 on K_{10}. Hence we have

$$\alpha \sim A_2' : h', \qquad \bar{\alpha} \sim \bar{A}' : h', \qquad \beta_2 \sim B_8' : h'^2 + \rho_6', \qquad \gamma_3 \sim \Gamma_{14}' : h'^3 + \rho_6'^2,$$

$$s_1 \sim s_9' : h'^2 + \rho_6',$$

and in the involution I

$$s_1 \sim s_{36} : \gamma_3^{14} + \alpha^7 + \bar{\alpha}^7 + \beta_2^8, \qquad \alpha \sim A_8 : \gamma_3^3 + \alpha^2 + \bar{\alpha} + \beta_2^2,$$

$$\bar{\alpha} \sim \bar{A}_8 : \gamma_3^3 + \alpha + \bar{\alpha}^2 + \beta_2^2, \qquad \beta_2 \sim B_{32} : \gamma^{12} + \alpha^6 + \bar{\alpha}^6 + \beta_3^8,$$

$$\gamma_3 \sim \Gamma_{56} : \gamma_3^{22} + \alpha^{11} + \bar{\alpha}^{11} + \beta_2^{12}, \ \rho_{32} \sim R_4 : \gamma_3^2 + \beta_2, \ h_4 \sim H_2 : \gamma_3 + \alpha + \bar{\alpha}.$$

34. **Case E.** The basis curve is a quintic β_5 of genus 1, meeting γ_3 in 7 points. There are three points P. Here

$$s_1' \sim s_5 : \gamma_3^2 + \beta_5 + 3P,$$

$$c_1' \sim c_8, \qquad p = 1; \qquad [c_8, \gamma_3] = 13, \qquad [c_8, \beta_5] = 9.$$

* The arrangement of the double points on the singular tangent planes can be determined at once from the preceding discussion of the curves on the composite surfaces of the web. While L_4' is the same in this as in the preceding case, the distribution of the fundamental curves in (x) is quite different.

The bisecants of γ_3 which meet β_5 lie on a surface $R_6 : \gamma_3^3 + \beta_5$ and have for images the points of a curve ρ_7', touching L_4' in 14 points R', images of the 14 generators r of R_6 lying on K_{10}.

The web contains three composite quintics of which one component is a quadric through γ_3 and two of the points P; it also contains three nodal quintics. Hence L_4' has 6 singular tangent planes. There are $3 \cdot$ lines g, bisecants of γ_3 through each point P; one trisecant d of β_5 meets γ_3. The three nodal quintics have a common quintic curve and meet in pairs in one of three cubics; there are 6 conics meeting γ_3 in 3 points and β_5 in 3 points; two pass through each P.

The surface L_4' has therefore 14 double points and is the focal surface of a line congruence of order 2 and class 4. We have

$$s_1 \sim s_8' : \rho_7', \qquad \beta_5 \sim B_9' : \rho_7', \qquad \gamma_3 \sim \Gamma_{13}' : \rho_7'^2,$$

and in the involution I

$$s_1 \sim s_{33} : \gamma_3^{33} + \beta_5^7, \qquad \beta_5 \sim B_{39} : \gamma_3^{15} + \beta_5^9,$$

$$\gamma_3 \sim \Gamma_{53} : \gamma_3^{21} + \beta_5^{11}, \qquad \rho_{32} \sim R_6 : \gamma_3^3 + \beta_5.$$

35. **Case F.** The quintic basis curve consists of two bisecants α, $\bar{\alpha}$ of γ_3 and a cubic β_3 meeting γ_3 in 4 points; there are 2 basis points P. We therefore have

$$s_1' \sim s_5 : \gamma_3^2 + \alpha + \bar{\alpha} + \beta_3 + 2P,$$

$$c_1' \sim c_8, \qquad p = 1; \qquad [c_8, \gamma_3] = 12, \qquad [c_8, \alpha] = 2, \qquad [c_8, \beta_3] = 8.$$

The quadric H_2 through γ_3, α, and $\bar{\alpha}$ with a cubic of the pencil through γ_3, β_3 and the points P make up a composite quintic of the web. Hence the image of H_2 is a line h' whose image in (x) is H_2 and the cubic curve h_3, common to the pencil of cubic surfaces. There are 6 singular tangent planes to L_4' images of the four composite quintics of the web; one component is a quadric through γ_3, α or $\bar{\alpha}$ and one point P; the other two images are the nodal quintics. The nodal quintics intersect in 2 fundamental quartics. There is a bisecant g of γ_3 through each point P, and there are four lines d meeting γ_3, α, $\bar{\alpha}$ and β_3.

There are 4 fundamental conics, each passing through one point P, meeting β_3 in 3 points, γ_3 in 2, and meeting each bisecant α. Two conics constitute the intersection of two of the cubics in the composite quintics. Finally, there are 2 cubics, each meeting γ_3 in 4 points, β_3 in 4 points, each bisecant in one point, and passing through one point P. The 2 quartics, the 2 lines g, the 4 lines d, the 4 conics and the 2 cubics have for images the 14 double points of L_4', which is the same surface as in Case E.

The bisecants of γ_3 which meet β_3 lie on a surface $R_4 : \gamma_3^2 + \beta_3$ and have for images the points of a curve ρ_5' which is tangent to L_4' at 10 points R'; these points of contact are images of the 10 generators of R_4 which lie on K_{10}. Hence we may write

$$s_1 \sim s_8' : h'^2 + \rho_5',$$

$$\alpha \sim A_2' : h', \qquad \overline{\alpha} \sim \overline{A}_2' : h', \qquad \beta_3 \sim B_8' : h'^2 + \rho_5', \qquad \gamma_3 \sim \Gamma_{12}' : h'^3 + \rho_5',$$

and in the involution I

$$s_1 \sim s_{31} : \gamma_3^{12} + \alpha^6 + \overline{\alpha}^6 + \beta_3^7, \qquad \alpha \sim A_8 : \gamma_3^3 + \alpha^2 + \overline{\alpha} + \beta_3^2,$$

$$\overline{\alpha} \sim \overline{A}_8 : \gamma_3^3 + \alpha + \overline{\alpha}^2 + \beta_3^2, \qquad \gamma_3 \sim \Gamma_{46} : \gamma_3^{18} + \alpha^9 + \overline{\alpha}^9 + \beta_3^{10},$$

$$\beta_3 \sim B_{32} : \gamma_3^{12} + \alpha^6 + \overline{\alpha}^6 + \beta_3^8, \quad h_3 \sim H_2 : \gamma_3 + \alpha + \overline{\alpha}, \quad \rho_{21} \sim R_4 : \gamma_3^3 + \beta_3.$$

36. **Case G.** Here we have a sextic basis curve β_6 of genus 1, meeting γ_3 in 9 points; there are 2 basis points P. Hence

$$s_1' \sim s_5 : \gamma_3^2 + \beta_6 + 2P,$$

$$c_1' \sim c_7, \qquad p = 1; \qquad [c_7, \gamma_3] = 11, \qquad [c_7, \beta_6] = 9.$$

The bisecants of γ_3 which meet β_6 lie on $R_6 : \gamma_3^3 + \beta_6$ and have for images the points of the curve ρ_6' which is tangent to L_4' in 12 points R', images of the 12 generators of R_6 on K_{10}. The web contains one composite quintic, one component being the quadric through γ_3 and the points P; there are two nodal quintics. The surface L_4' has therefore 3 singular tangent planes. There are 2 lines g, bisecants of γ_3, through each point P; there are also 3 trisecants of β_6 which meet γ_3.

The two nodal quintics meet in a fundamental quartic and cubic. Through each point P pass three fundamental conics meeting γ_3 and β_6 each in 3 points. Hence the surface L_4' has 13 double points and is the focal surface of a line congruence of order 2 and class 5.*

We then have

$$s_1 \sim s_7' : \rho_6', \qquad \beta_6 \sim B_9' : \rho_6', \qquad \gamma_3 \sim \Gamma_{11}' : \rho_6'^2,$$

and in the involution I

$$s_1 \sim s_{28} : \gamma_3^{11} + \beta_6^6, \qquad \gamma_3 \sim \Gamma_{43} : \gamma_3^{17} + \beta_6^9,$$

$$\beta_6 \sim B_{39} : \gamma_3^{15} + \beta_6^9, \qquad \rho_{21} \sim R_6 : \gamma_3^3 + \beta_6.$$

37. **Case H.** The basis sextic consists of 2 bisecants α, $\overline{\alpha}$ of γ_3 and a quartic β_4 meeting γ_3 in 6 points; there is one point P. Hence

$$s_1' \sim s_5 : \gamma_3^2 + \alpha + \overline{\alpha} + \beta_4 + P,$$

$$c_1' \sim c_7, \qquad p = 1; \qquad [c_7, \gamma_3] = 10, \qquad [c_7, \alpha] = 2, \qquad [c_7, \beta_4] = 8.$$

* Kummer, l. c., pp. 88–94.

The bisecants of γ_3 which meet β_4 lie on a surface $R_4 : \gamma_3^2 + \beta_4$ and have for images the points of a curve ρ_4' which touches L_4' in 8 points R'; the points of contact are the images of the 8 generators of R_4 which lie on K_{10}. The quadric $H_2 : \gamma_3 + \alpha + \overline{\alpha}$ and a cubic surface of the pencil through γ_3, β_4, and P make up a composite quintic of the web. Hence the image of H_2 is a line h' whose image in (x) is H_2 and the conic h_2 common to the cubics of the pencil.

There are 2 composite quintics with one component a quadric through γ_3, α or $\overline{\alpha}$ and P; there is one nodal quintic. The line g is the bisecant of γ_3 through P. Three lines meet β_4 twice and meet γ_3 and α or $\overline{\alpha}$; the composite quintics intersect each other and each intersects the nodal quintic in a fundamental conic; two of the conics pass through P, meet β_4 in 2 points, α_3 in 3 points, and α or $\overline{\alpha}$ in one; the third meets α and $\overline{\alpha}$, β_4 in 2 points, and γ_3 in 3 points. There are 3 fundamental cubics meeting α, $\overline{\alpha}$, γ_3 in 4 points, β_4 in 4 and passing through P.

The surface L_4' has the same form as in the last preceding case.

We have then

$$s_1 \sim s_7' : h'^2 + \rho_4', \qquad \gamma_3 \sim \Gamma_{10}' : h'^2 + \rho_4'^2, \qquad \alpha \sim A_2' : h', \qquad \beta_4 \sim B_8' : \rho_4',$$

and in the involution I

$$s_1 \sim s_{26} : \gamma_3^{10} + \alpha^5 + \overline{\alpha}^5 + \beta_4^6, \qquad \gamma_3 \sim \Gamma_{36} : \gamma_3^{14} + \alpha^7 + \overline{\alpha}^7 + \beta_4^8,$$

$$\alpha \sim A_8 : \gamma_3^8 + \alpha^2 + \overline{\alpha} + \beta_4^2, \qquad \overline{\alpha} \sim \overline{A}_8 : \gamma_3^3 + \alpha + \overline{\alpha}^2 + \beta_4^2,$$

$$\beta_4 \sim B_{32} : \gamma_3^{12} + \alpha^8 + \overline{\alpha}^8 + \beta_4^8, \quad h_2 \sim H_2 : \gamma_3 + \alpha + \overline{\alpha}, \quad \rho_{12} \sim R_4 : \gamma_3^2 + \beta_4 .$$

38. **Case I.** The basis curve is a β_7 of genus 1, with no five-fold secants, meeting γ_3 in 11 points; there is one basis point P. Here

$$s_1' \sim s_5 : \gamma_3^2 + \beta_7 + P,$$

$$c_1' \sim c_6, \qquad p = 1; \qquad [c_6, \gamma_3] = 9, \qquad [c_6, \beta_7] = 9.$$

The bisecants of γ_3 which meet β_7 lie on a surface $R_6 : \gamma_3^3 + \beta_7$ and have for images the points of a curve ρ_5', tangent to L_4' at 10 points, images of the 10 generators r common to R_6 and K_{10}.

There is one singular tangent plane of L_4, image of the nodal quintic of the web. There is one line g, bisecant of γ_3 from P, and 6 lines d, trisecants of β_7 meeting γ_3, also 4 fundamental conics and one cubic. Hence L_4' has 12 double points. It is the focal surface of a line congruence of order 2 and class 6.*

Hence

$$s_1 \sim s_6' : \rho_5', \qquad \gamma_3 \sim \Gamma_9' : \rho_5'^2, \qquad \beta_7 \sim B_9' : \rho_5',$$

* Kummer, l. c., pp. 102–107.

and in the involution I

$$s_1 \sim s_{23} : \gamma_3^9 + \beta_7^5, \qquad \gamma_3 \sim \Gamma_{33} : \gamma_3^{13} + \beta_7^7,$$

$$\beta_7 \sim B_{39} : \gamma_3^{15} + \beta_7^9, \qquad \rho_{12} \sim R_6 : \gamma_3^3 + \beta_7.\text{*}$$

39. **Case J.** The basis curve consists of 2 bisecants $\alpha, \overline{\alpha}$ of γ_3 and a rational quintic β_5 meeting γ_3 in 8 points, hence

$$s_1' \sim s_5 : \gamma_3^2 + \alpha + \overline{\alpha} + \beta_5,$$

$$c_1' \sim c_6, \qquad p = 1; \qquad [c_6, \gamma_3] = 8, \qquad [c_6, \alpha] = 2, \qquad [c_6, \beta_5] = 8.$$

The quadric $H_2 : \gamma_3 + \alpha + \overline{\alpha}$ and a cubic of the pencil through β_5 and γ_3 make up a composite quintic of the web. Hence the image of H_2 is a line h', whose complete image in (x) consists of H_2 and the line h, common to the cubics of the pencil. The bisecants of γ_3 which meet β_5 lie on a surface $R_4 : \gamma_3^2 + \beta_5$ and have for images the points of a cubic ρ_3' tangent to L_4' at 6 points, images of the 6 generators r common to R_4 and to K_{10}. We have therefore

$$s_1 \sim s_6' : h'^2 + \rho_3', \qquad \gamma_3 \sim \Gamma_8' : h'^3 + \rho_3'^2,$$

$$\alpha \sim A_2^1 : h', \qquad \beta_5 \sim B_8' : h'^2 + \rho_3',$$

and in the involution I

$$s_1 \sim s_{21} : \gamma_3^8 + \alpha^4 + \overline{\alpha}^4 + \beta_5^5, \qquad \gamma_3 \sim \Gamma_{20} : \gamma_3^{10} + \alpha^5 + \overline{\alpha}^5 + \beta_5^6,$$

$$\beta_5 \sim B_{32} : \gamma_3^{13} + \alpha^6 + \overline{\alpha}^6 + \beta_5^8, \qquad \alpha \sim A_8 : \gamma_3^3 + \alpha^2 + \overline{\alpha}^2 + \beta_5^2,$$

$$\rho_5 \sim R_4 : \gamma_3^3 + \beta_5, \qquad h \sim H_2 : \gamma_3 + \alpha + \overline{\alpha}.$$

There are 8 fundamental lines and 4 fundamental conics. The surface L_4' is the focal surface of a line congruence of order 2 and class 6, without singular planes.†

40. **Quintics with a double quartic curve.** The curve γ_4 must be of genus 1. By Art. 16 we have

$$p' + 3s = 6m' - 2, \qquad \xi + s = m' + 5,$$

the possible solutions of which are

	m'	p'	s	ξ
A......	3	-2	6	2
B......	4	-2	8	1
C......	5	-2	10	0

* If β_7 lies on a cubic surface (Noether, l. c., p. 91), this surface contains γ_3 and there is a pencil of composite quintics. The corresponding involution differs somewhat from the type just obtained.

† Kummer, l. c., pp. 94–102.

41. **Case A.** The simple basis curve consists of three bisecants of γ_4; there are 2 basis points P. By means of a cubic transformation having $\gamma_4 + 2\alpha$ for a basis sextic of genus 3, the web of quintics can be transformed into a web of cubics having a quartic curve of genus 1, a bisecant, and 2 basis points for basis elements. The quartic and its bisecant constitute a quintic of genus 2, hence this is included as a particular case of that discussed in Article 9.

42. **Case B.** By proceeding as in the last preceding case, this is at once reducible to that of Article 10.

43. **Case C.** The simple basis curve consists of a bisecant α of γ_4 and of 2 conics β_2, $\overline{\beta}_2$ each meeting γ_4 in 4 points. We have therefore

$$s_1' \sim s_5 : \gamma_4^2 + \alpha + \beta_2 + \overline{\beta}_2,$$

$$c_1' \sim c_4, \quad p = 1; \quad [c_4, \alpha] = 2, \quad [c_4, \gamma_4] = 6, \quad [c_4, \beta_2] = 2.$$

The quadric $R_2 : \gamma_4 + \alpha$ and a cubic of the pencil through γ_4, β_2 and $\overline{\beta}_2$ make up a composite quintic of the web. The image of R_2 is therefore a line ρ' having for image in (x) the quadric R_2 and the line ρ common to the cubics of the pencil. The quadric $H_2 : \gamma_4 + \beta_2$ and a cubic of the bundle through γ_4, $\overline{\beta}_2$, and α make up a composite quintic of the web. The image of H_2 is therefore a point P'. Similarly, the image of $\overline{H}_2 : \gamma_4 + \overline{\beta}_2$ is a point $\overline{P}'$. We have therefore

$$s_1 \sim s_4' : \rho' + P' + \overline{P}', \qquad \alpha \sim A_2' : \rho', \qquad \beta_2 \sim B_2' : P', \qquad \overline{\beta}_2 \sim \overline{B}_2' : \overline{P}',$$

$$\gamma_4 \sim \Gamma_6 : \rho'^2 + P'^2 + \overline{P}'^2,$$

and in the involution I

$$s_1 \sim s_{13} : \gamma_4^5 + \alpha^3 + \beta_2^3 + \overline{\beta}_2^3, \qquad \alpha \sim A_8 : \gamma_4^3 + \alpha^2 + \beta_2^2 + \overline{\beta}_2^2,$$

$$\beta_2 \sim B_8 : \gamma_4^3 + \alpha^2 + \beta_2^2 + \overline{\beta}_2^2, \qquad \overline{\beta}_2 \sim \overline{B}_8 : \gamma_4^3 + \alpha^2 + \beta_2^2 + \overline{\beta}_2^2,$$

$$\gamma_4 \sim \Gamma_{18} : \gamma_4^7 + \alpha^4 + \beta_2^4 + \overline{\beta}_2^4.$$

The line ρ' is tangent to L_4' at 2 points R', images of the 2 generators of R_2 on K_{10}. There are 8 fundamental lines.

44. **Conclusion.** This completes the consideration of involutions derivable from $(2, 1)$ correspondences using webs of surfaces of order not greater than 5. When surfaces of higher order are used, all the involutions associated with L' of order 4 are reducible to some type obtained in this paper.

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DIFFERENTIAL EQUATIONS CONTAINING ARBITRARY FUNCTIONS*

BY

GILBERT AMES BLISS

A recent study of the problem of computing differential corrections for a trajectory has led the writer to the present investigation of the properties of solutions of differential equations considered as functions of other arbitrary functions which enter into the differential equations themselves. In ballistics it is customary to compute first the coördinates of a trajectory, uninfluenced by wind or other disturbances, as solutions of the differential equations of motion. A new system of differential equations is then set up which accounts for the disturbances of the trajectory, and with the help of which corrections to the coördinates of the projectile on the original trajectory can be computed. The new equations involve arbitrary functions and the corrections found are themselves functions of these functions. The details of this situation will be described in another paper. In the following pages the results attained are of a more general character than is necessary for the ballistic problem, and it seems probable that they will be useful in many other connections.

The differential equations here considered have the form

$$(1)\qquad \frac{dx^{(i)}}{d\tau} = f^{(i)}(\tau, x^{(1)}, \cdots, x^{(p)}) \qquad (i = 1, 2, \cdots, p),$$

and it is supposed that a particular solution

$$(2)\qquad x^{(i)} = u^{(i)}(\tau) \qquad (i = 1, \cdots, p;\ \tau_1 \leqq \tau \leqq \tau_2)$$

for a particular set of functions $f^{(i)}$ is known. When the functions $f^{(i)}$ themselves are allowed to vary the solutions of these equations are functions $x^{(i)}[\tau, \tau_0, x_0, f]$ of the variable τ, of the coördinates τ_0 and

$$x_0 = (x_0^{(i)}, \cdots, x_0^{(p)})$$

of a prescribed initial point (τ, x_0) of the curve, and of the functions $f^{(i)}$ in the second members of the equations. In Sections 1 and 2 below the character of these solutions in a neighborhood of the particular solution (2) are considered.

* Presented to the Society, September, 1919.

41. **Case A.** The simple basis curve consists of three bisecants of γ_4; there are 2 basis points P. By means of a cubic transformation having $\gamma_4 + 2\alpha$ for a basis sextic of genus 3, the web of quintics can be transformed into a web of cubics having a quartic curve of genus 1, a bisecant, and 2 basis points for basis elements. The quartic and its bisecant constitute a quintic of genus 2, hence this is included as a particular case of that discussed in Article 9.

42. **Case B.** By proceeding as in the last preceding case, this is at once reducible to that of Article 10.

43. **Case C.** The simple basis curve consists of a bisecant α of γ_4 and of 2 conics β_2, $\overline{\beta}_2$ each meeting γ_4 in 4 points. We have therefore

$$s_1' \sim s_5 : \gamma_4^2 + \alpha + \beta_2 + \overline{\beta}_2,$$

$$c_1' \sim c_4, \quad p = 1; \quad [c_4, \alpha] = 2, \quad [c_4, \gamma_4] = 6, \quad [c_4, \beta_2] = 2.$$

The quadric $R_2 : \gamma_4 + \alpha$ and a cubic of the pencil through γ_4, β_2 and $\overline{\beta}_2$ make up a composite quintic of the web. The image of R_2 is therefore a line ρ' having for image in (x) the quadric R_2 and the line ρ common to the cubics of the pencil. The quadric $H_2 : \gamma_4 + \beta_2$ and a cubic of the bundle through γ_4, $\overline{\beta}_2$, and α make up a composite quintic of the web. The image of H_2 is therefore a point P'. Similarly, the image of $\overline{H}_2 : \gamma_4 + \overline{\beta}_2$ is a point $\overline{P}'$. We have therefore

$$s_1 \sim s_1' : \rho' + P' + \overline{P}', \qquad \alpha \sim A_2' : \rho', \qquad \beta_2 \sim B_2' : P', \qquad \overline{\beta}_2 \sim \overline{B}_2' : \overline{P}',$$

$$\gamma_4 \sim \Gamma_6 : \rho'^2 + P'^2 + \overline{P}'^2,$$

and in the involution I

$$s_1 \sim s_{13} : \gamma_4^5 + \alpha^3 + \beta_2^3 + \overline{\beta}_2^3, \qquad \alpha \sim A_8 : \gamma_4^3 + \alpha^2 + \beta_2^2 + \overline{\beta}_2^2,$$

$$\beta_2 \sim B_8 : \gamma_4^3 + \alpha^2 + \beta_2^2 + \overline{\beta}_2^2, \qquad \overline{\beta}_2 \sim \overline{B}_8 : \gamma_4^3 + \alpha^2 + \beta_2^2 + \overline{\beta}_2^2,$$

$$\gamma_4 \sim \Gamma_{18} : \gamma_4^7 + \alpha^4 + \beta_2^4 + \overline{\beta}_2^4.$$

The line ρ' is tangent to L_4' at 2 points R', images of the 2 generators of R_2 on K_{10}. There are 8 fundamental lines.

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